Vectors and Scalars

AP Physics B

Scalar

A **SCALAR** is ANY quantity in physics that has MAGNITUDE, but NOT a direction associated with it.

Magnitude – A numerical value with units.

Scalar Example	Magnitude
Speed	20 m/s
Distance	10 m
Age	15 years
Heat	1000 calories

Vector

A **VECTOR** is ANY quantity in physics that has BOTH MAGNITUDE and DIRECTION.

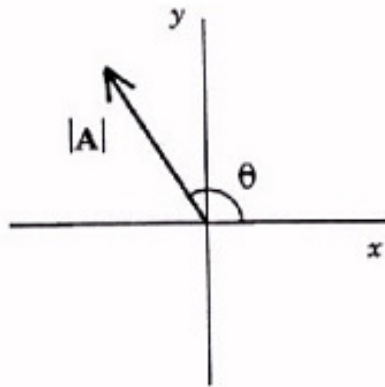| Vector	Magnitude & Direction
Velocity	20 m/s, N
Acceleration	10 m/s/s, E
Force	5 N, West

$$\vec{v}, \vec{x}, \vec{a}, \vec{F}$$

Vectors are typically illustrated by drawing an ARROW above the symbol. The arrow is used to convey direction and magnitude.

Q? How do you describe the fly's movement?

$\vec{A}$

1 foot

45°

This arrow is a physical representation of the displacement of the fly.

It is a **vector**, denoted by the arrow over its symbol.

The symbols $|\vec{A}|$ and A represent the magnitude of the vector $\vec{A}$.

A! It went **one foot** in a straight path at an angle of **45 degrees**.

distance   direction
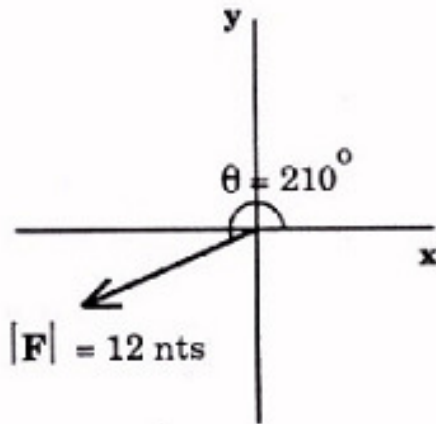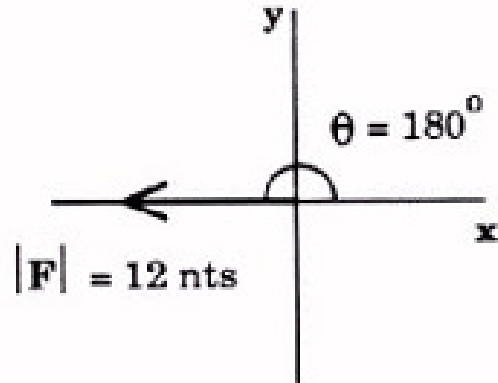
Polar Notation

Polar notation defines a vector by designating the vector's magnitude |A| and angle θ relative to the +x axis. Using that notation the vector is written:

$$A = |A| \angle \theta$$
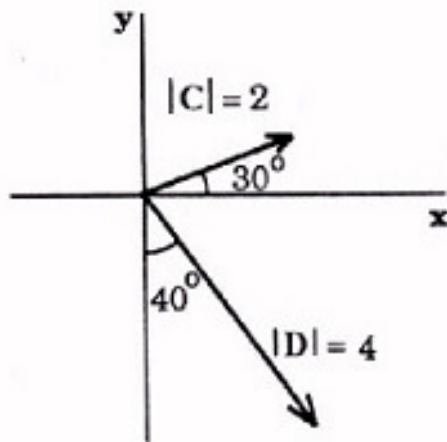

In this picture we have a force vector with magnitude 12 Newtons oriented at 210 degrees with the + x axis. It would be characterized as **F = 12 ∠ 210**

Polar Notation

In this picture we have a force vector of 12 Newtons oriented along the -x axis. However, polar notation is relative to the + x axis. Therefore, it would be characterized by
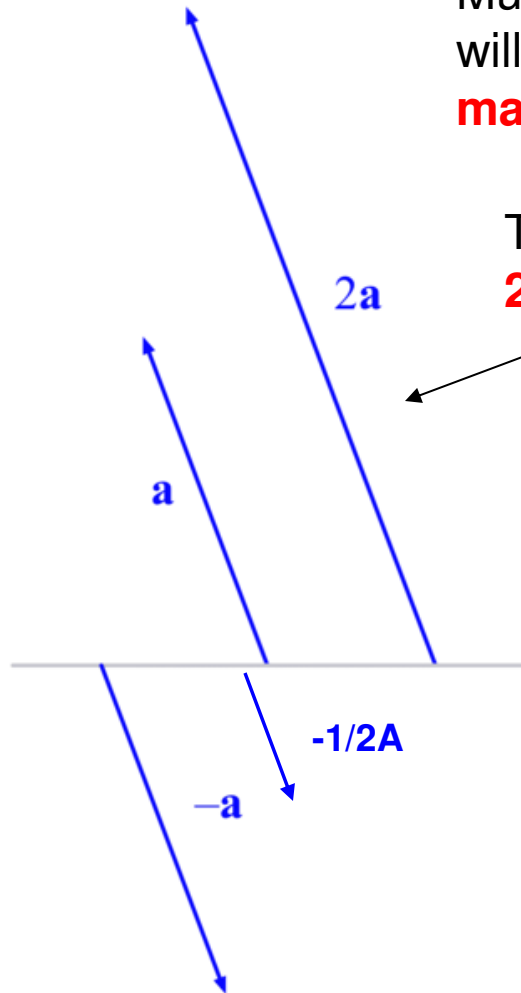
F = 12 < 180

In this last picture we have 2 vectors. They are characterized by:

C = 2 < 30

D = 4 < - 50 or D = 4 < 310

Scalar Multiplication

Multiplying a vector by a scalar will ONLY CHANGE its **magnitude**.

Multiplying a vector by "-1" does not change the magnitude, but it does reverse it's direction or in a sense, it's angle.

Thus if **A = 12 < 105**, Then **2A = 24 < 105**
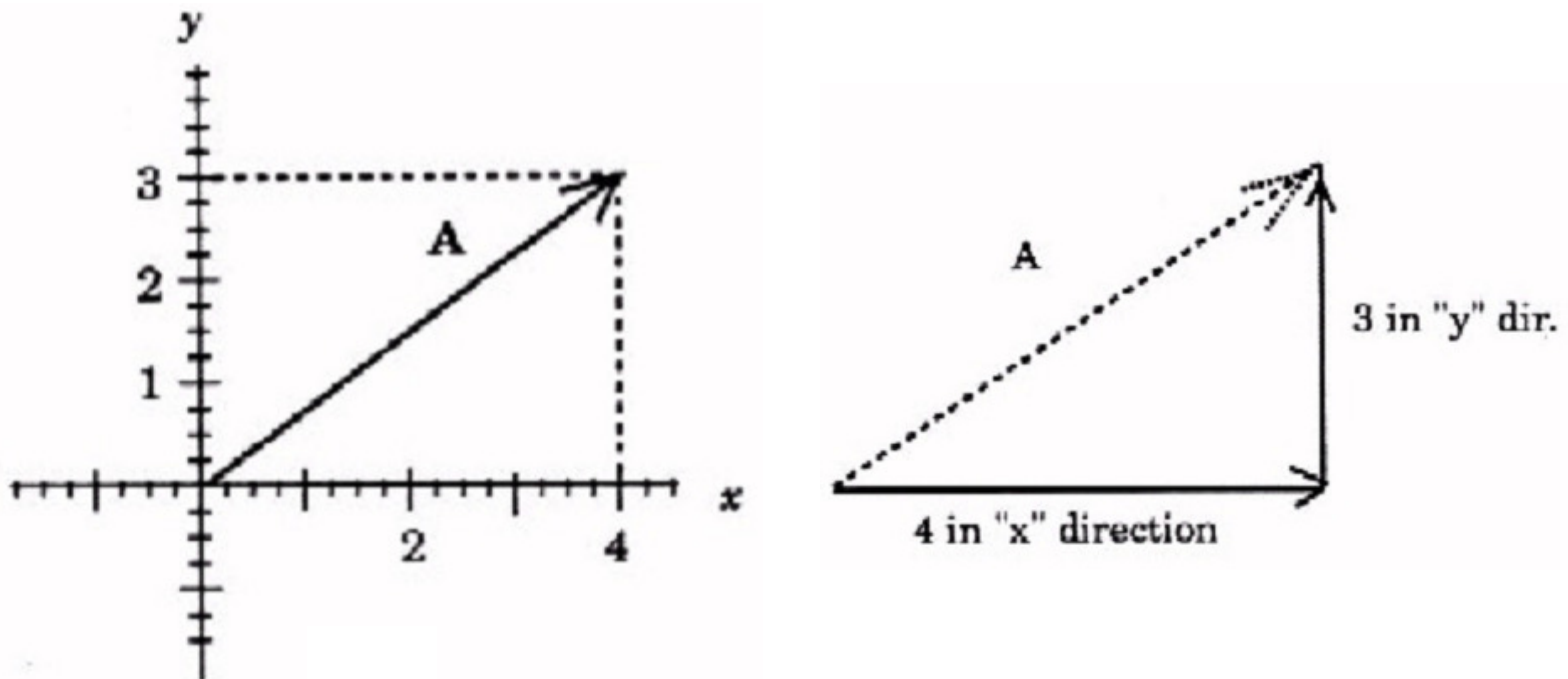
2a

a

-1/2A

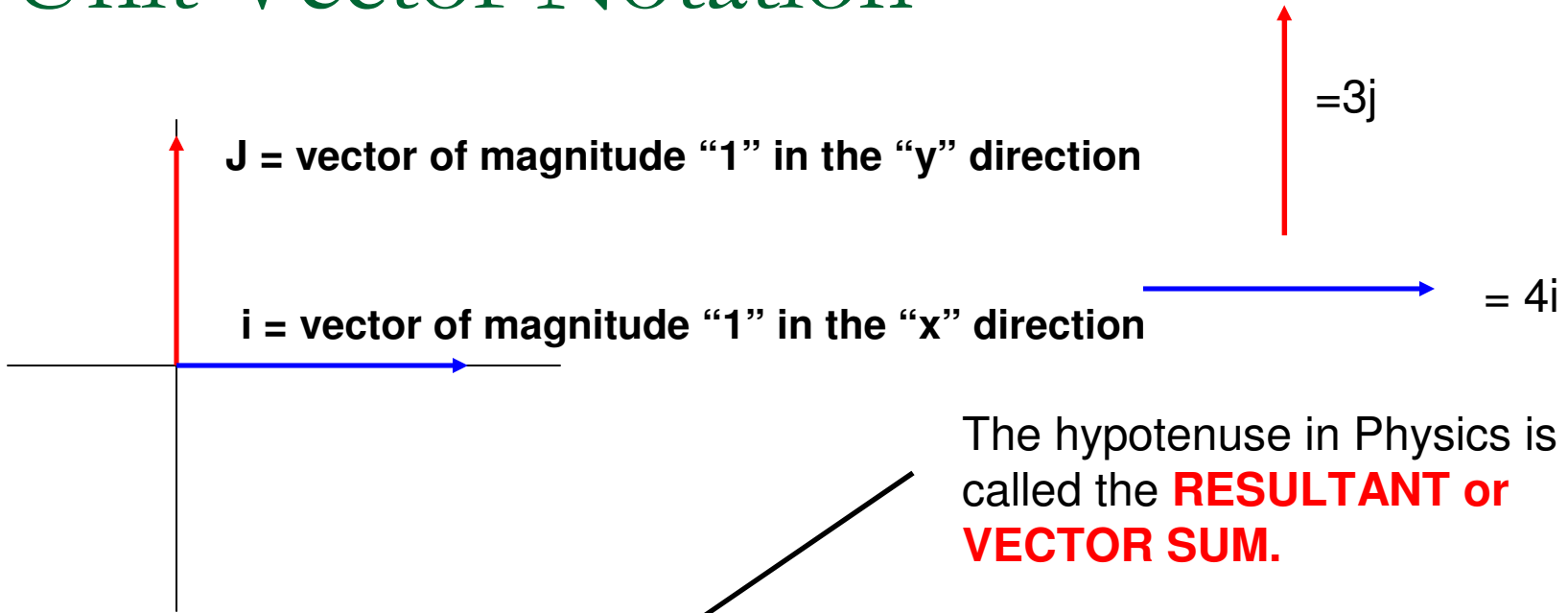−a

Thus if **A = 12 < 105**, then **-A = 12 < 285**

If **A = 12 < 105**, then **(-1/2)A = 6 < 285**

Unit Vector Notation

An effective and popular system used in engineering is called **unit vector notation**. It is used to denote vectors with an x-y Cartesian coordinate system.
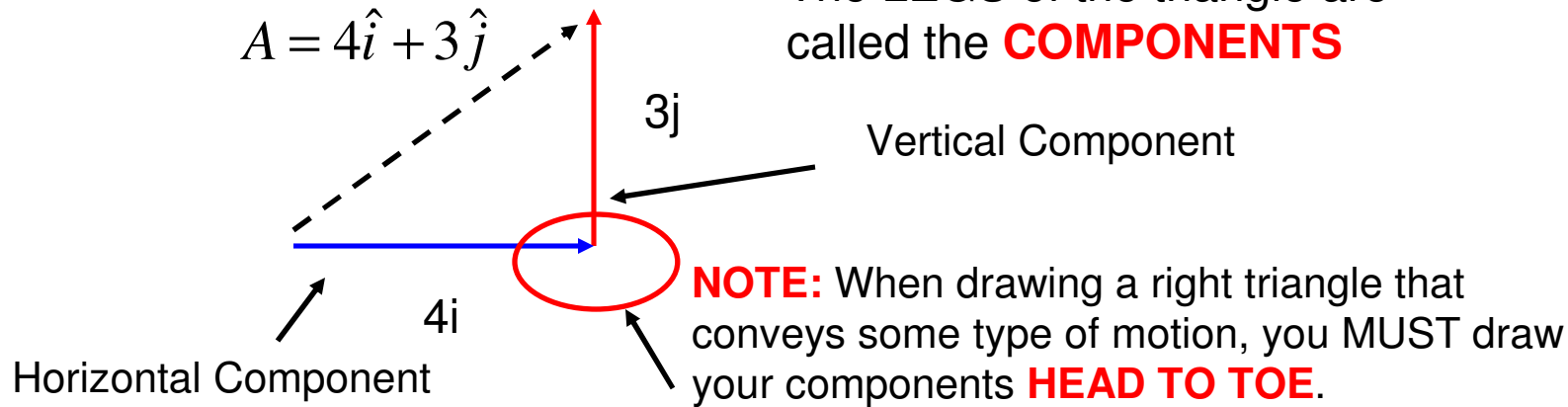
Unit Vector Notation

J = vector of magnitude "1" in the "y" direction

=3j

i = vector of magnitude "1" in the "x" direction

= 4i

The hypotenuse in Physics is called the **RESULTANT or VECTOR SUM.**

$$A = 4\hat{i} + 3\hat{j}$$

The LEGS of the triangle are called the **COMPONENTS**

3j

Vertical Component

4i

Horizontal Component

NOTE: When drawing a right triangle that conveys some type of motion, you MUST draw your components **HEAD TO TOE**.
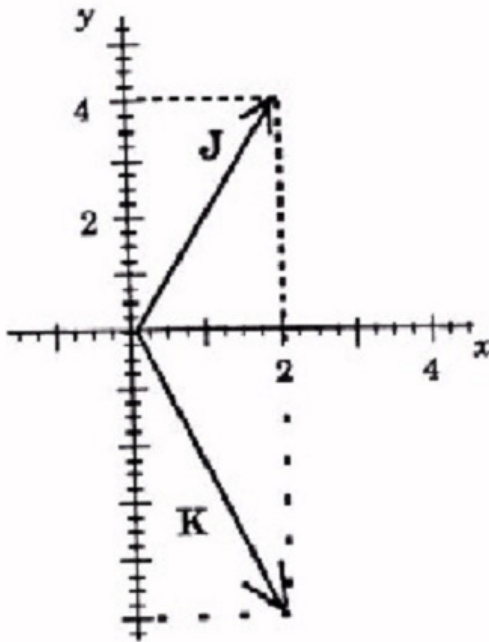
Unit Vector Notation

$\hat{i}$ - unit vector $= 1$ in the $+ \, x$ direction

$\hat{j}$ - unit vector $= 1$ in the $+ \, y$ direction

$\hat{k}$ - unit vector $= 1$ in the $+ \, z$ direction

The proper terminology is to use the "hat" instead of the arrow. So we have i-hat, j-hat, and k-hat which are used to describe any type of motion in 3D space.

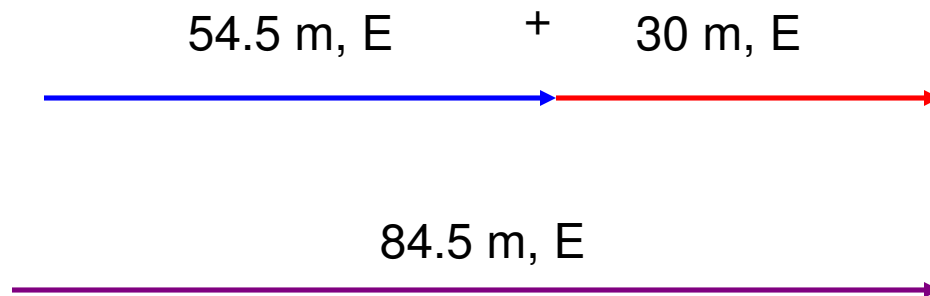**How would you write vectors J and K in unit vector notation?**

$$J = 2\hat{i} + 4\hat{j}$$

$$K = 2\hat{i} - 5\hat{j}$$

Applications of Vectors

VECTOR ADDITION – If 2 similar vectors point in the SAME direction, add them.

- **Example: A man walks 54.5 meters east, then another 30 meters east. Calculate his displacement relative to where he started?**
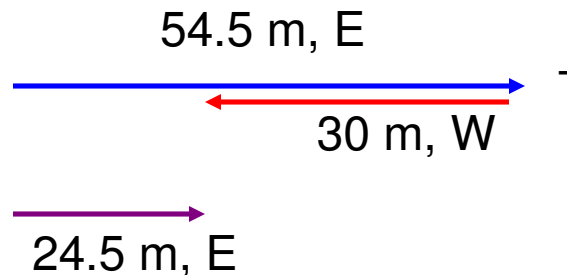
54.5 m, E   +   30 m, E

84.5 m, E

Notice that the SIZE of the arrow conveys MAGNITUDE and the way it was drawn conveys DIRECTION.

Applications of Vectors

**VECTOR SUBTRACTION** - If 2 vectors are going in opposite directions, you **SUBTRACT**.

- **Example: A man walks 54.5 meters east, then 30 meters west. Calculate his displacement relative to where he started?**

54.5 m, E

-

30 m, W

24.5 m, E

Non-Collinear Vectors

When 2 vectors are perpendicular, you must use the Pythagorean theorem.

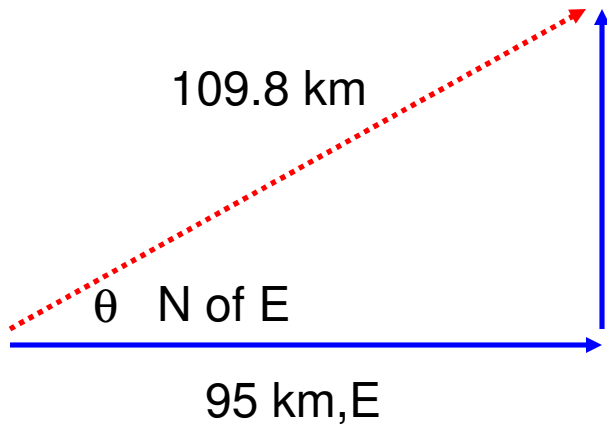A man walks 95 km, East then 55 km, north. Calculate his RESULTANT DISPLACEMENT.

55 km, N

95 km, E
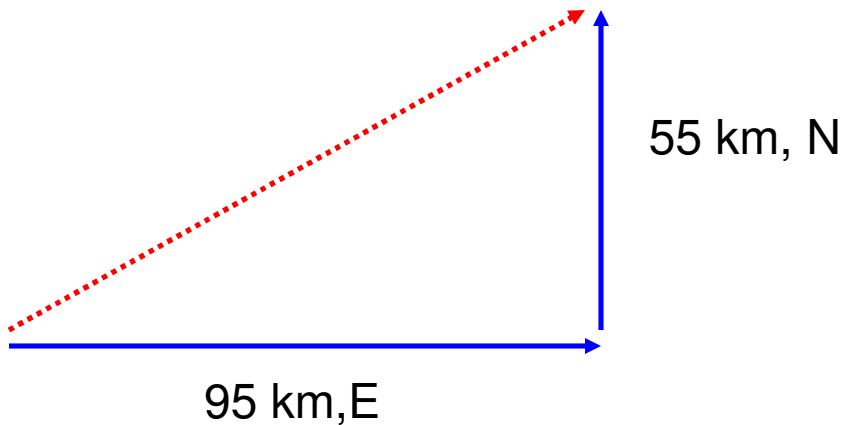
$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2}$$

$$c = \text{Resultant} = \sqrt{95^2 + 55^2}$$

$$c = \sqrt{12050} = 109.8 \; km$$

BUT…..what about the VALUE of the angle???

Just putting North of East on the answer is NOT specific enough for the direction. We MUST find the VALUE of the angle.

109.8 km

55 km, N

θ N of E

95 km,E

To find the value of the angle we use a Trig function called TANGENT.

$$Tan\,\theta = \frac{opposite\ side}{adjacent\ side} = \frac{55}{95} = 0.5789$$

$$\theta = Tan^{-1}(0.5789) = 30°$$

So the COMPLETE final answer is :
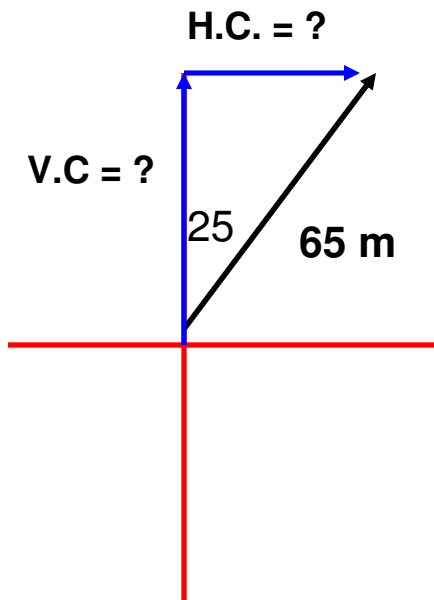
$$109.8km\,@\,30°\,NofE$$

$$109.8 < 30$$

$$95\hat{i}\,m + 55\hat{j}\,m$$

What if you are missing a component?

Suppose a person walked 65 m, 25 degrees East of North. What were his horizontal and vertical components?

H.C. = ?

V.C = ?

25

65 m

The goal: **ALWAYS MAKE A RIGHT TRIANGLE!**

To solve for components, we often use the trig functions sine and cosine.

$$\text{cosine}\,\theta = \frac{adjacent\ side}{hypotenuse} \qquad \text{sine}\,\theta = \frac{opposite\ side}{hypotenuse}$$

$$adj = hyp\cos\theta \qquad\qquad opp = hyp\sin\theta$$

$$adj = V.C. = 65\cos 25 = 58.91m, N\ or\ 58.91\hat{j}\,m$$

$$opp = H.C. = 65\sin 25 = 27.47m, E\ or\ 27.47\hat{i}\,m$$

Example

A bear, searching for food wanders 35 meters east then 20 meters north. Frustrated, he wanders another 12 meters west then 6 meters south. Calculate the bear's displacement.

23 m, E

12 m, W
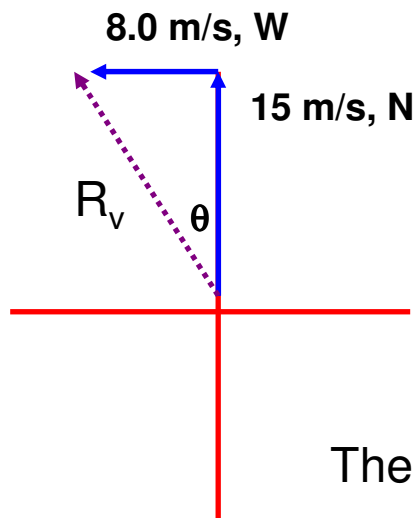
6 m, S

20 m, N

35 m, E

R

14 m, N

14 m, N

23 m, E

θ

$$R = \sqrt{14^2 + 23^2} = 26.93m$$

$$Tan\theta = \frac{14}{23} = .6087$$

$$\theta = Tan^{-1}(0.6087) = 31.3°$$
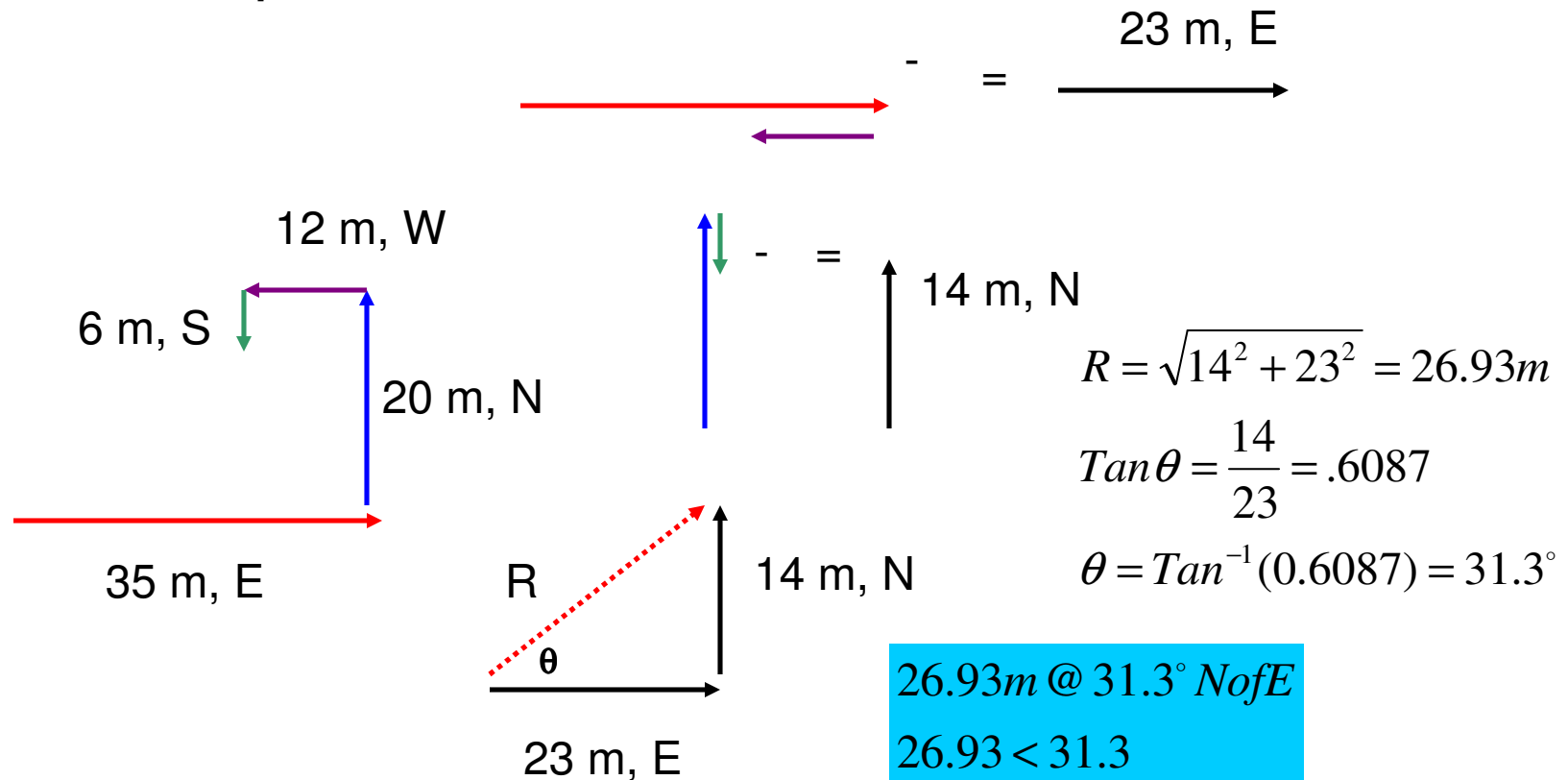
$$26.93m \ @ \ 31.3° \ NofE$$

$$26.93 < 31.3$$

$$23\hat{i} \ m + 14\hat{j} \ m$$

The Final Answer:

Example

A boat moves with a velocity of 15 m/s, N in a river which flows with a velocity of 8.0 m/s, west. Calculate the boat's resultant velocity with respect to due north.

$$R_v = \sqrt{8^2 + 15^2} = 17 \, m/s$$

8.0 m/s, W

15 m/s, N

R_v θ

$$Tan\,\theta = \frac{8}{15} = 0.5333$$

$$\theta = Tan^{-1}(0.5333) = 28.1°$$

The Final Answer :

$$17 \, m/s \, @ \, 28.1° \, WofN$$
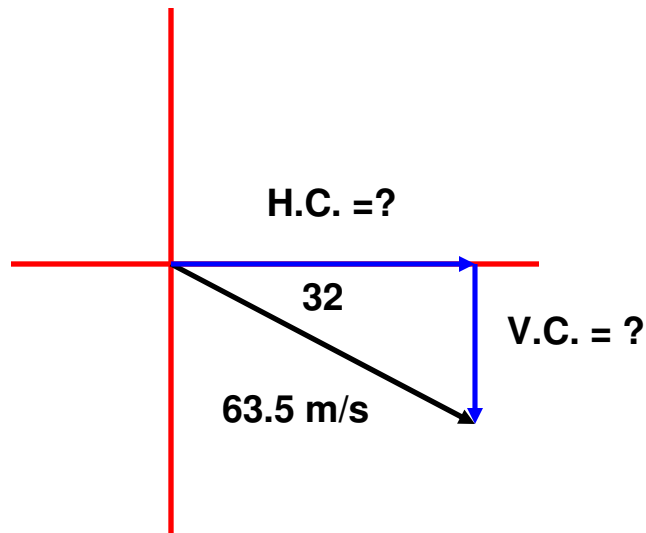
$$17 \, m/s < 118.1°$$

$$-8\hat{i} \, m/s + 15\hat{j} \, m/s$$

Example

A plane moves with a velocity of 63.5 m/s at 32 degrees South of East. Calculate the plane's horizontal and vertical velocity components.

H.C. =?

32

V.C. = ?

63.5 m/s

$$\text{cosine}\,\theta = \frac{adjacent\ side}{hypotenuse} \quad \text{sine}\,\theta = \frac{opposite\ side}{hypotenuse}$$

$$adj = hyp\cos\theta \qquad\qquad opp = hyp\sin\theta$$

$$adj = H.C. = 63.5\cos 32 = 53.85\,m/s, E\ or\ 53.85\hat{i}$$

$$opp = V.C. = 63.5\sin 32 = 33.64\,m/s, S\ or\ -33.64\,\hat{j}$$

The "Dot" Product (Vector Multiplication)

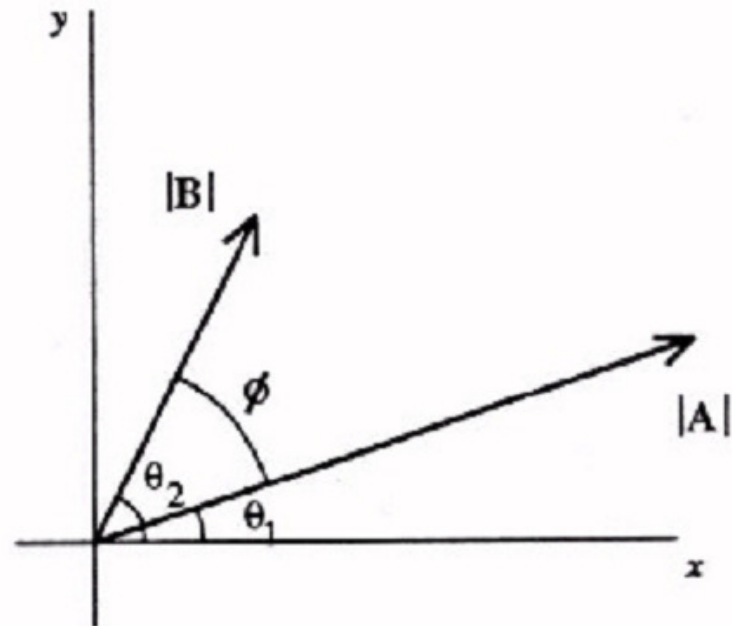Multiplying 2 vectors sometimes gives you a SCALAR quantity which we call the **SCALAR DOT PRODUCT**.

In polar notation consider 2 vectors:

A = |A| < θ₁ & **B = |B| < θ₂**

The dot product between A and B produces a SCALAR quantity. The magnitude of the scalar product is defined as:
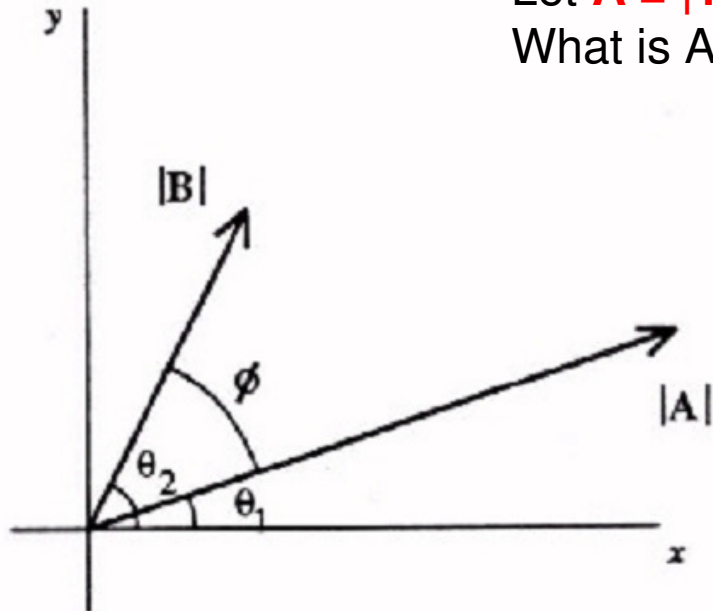
$$A \bullet B = |A||B|\cos\phi$$

Where θ is the NET angle between the two vectors. As shown in the figure.

The Scalar Dot Product

Let **A = |12| < 30**, Let **B = |5| < 65**
What is A "dot" B?

$$A \bullet B = |A||B| \cos \theta = |12||5| \cos 35$$

$$A \bullet B = 49.15$$

In unit vector notation, it looks a little different. Consider:

$$\mathbf{A = A_x i + A_y j + A_z k}$$
$$\mathbf{B = B_x i + B_y j + B_z k}$$

The "Dot" product between these is equal to:

$$A \bullet B = A_x B_x + A_y B_y + A_z B_z$$

The Scalar Dot Product

Example:

Let A = (3i - 4j - 5k)

Let B = (2i + 7j + 3k)

Therefore, A "dot" B = **(3)(2) + (-4)(7) +(-5)(3) = -37**

What is the **SIGNIFICANCE** of the dot product?

$$A \bullet B = |A||B|\cos \phi$$

The significance of the dot product

y

B sin Ø
(component of B
perpendicular to
line-of-A)

|B|

Ø

|A|

B cos Ø
(component of B
along line-of-A)

x
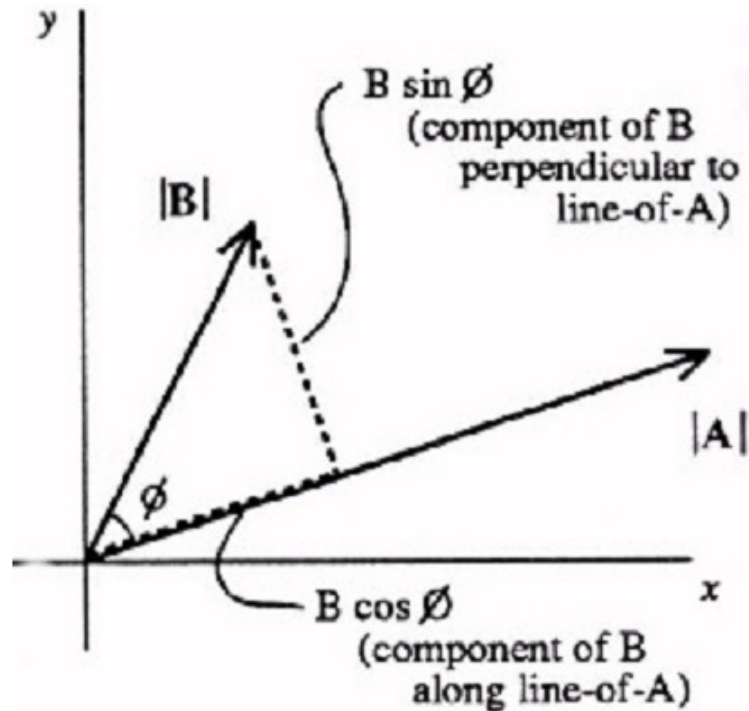
In this figure, vector B has been split into 2 components, one PARALLEL to vector A and one PERPENDICULAR to vector A. Notice that the component parallel to vector A has a magnitude of |B|Cos θ

THEREFORE when you find the DOT PRODUCT, the result is:

   i) The MAGNITUDE of one vector, in this case |A| and,

   ii) The MAGNITUDE of the 2nd vector's component that runs **parallel** to the first vector. (That is where the cosine comes from)

Dot Products in Physics
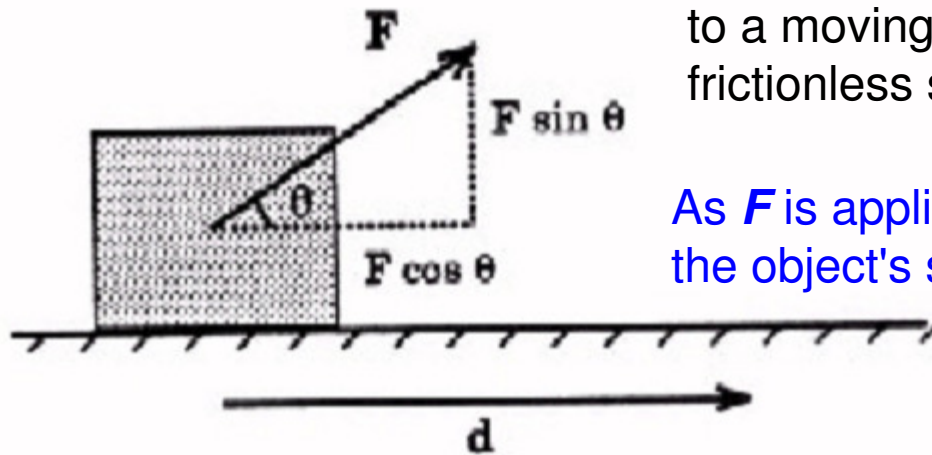
Consider this situation: A force **F** is applied to a moving object as it transverses over a frictionless surface for a displacement, **d**.

As **F** is applied to the object it will increase the object's speed!

But which part of **F** really causes the object to increase in speed?

It is **|F|Cos θ** ! Because it is **parallel** to the displacement **d**

In fact if you apply the dot product, you get **(|F|Cos θ)d,** which happens to be defined as "WORK" (check your equation sheet!)

$$A \bullet B = |A||B|\cos\theta$$

$$W = F \bullet x = |F||x|\cos\theta$$

Work is a type of energy and energy DOES NOT have a direction, that is why WORK is a scalar or in this case a SCALAR PRODUCT (AKA DOT PRODUCT).

Dot products using the TI-89

Here is where having the TI-89 is going to come if handy. You can actually enter vector into your calculator in UNIT VECTOR NOTATION. On your calculator we first need to clear any data that might be lurking in its memory. **Go to F6, then choose CLEAR a-z.**

This will wipe out any and all stored values that you have assigned to variable letters.

Now let's learn how to type a vector into the calculator in unit vector notation.

Suppose we had two vectors, one which represented the force and the other which represents the displacement as in our **WORK** example earlier.

$$\vec{F} = 3\hat{i} + 7\hat{j} - 2\hat{k}$$

$$\vec{d} = -5\hat{i} + 2\hat{j}$$

Dot products using the TI-89

You start off by using brackets, then separate each number by a comma, then close the brackets. Then hit the STO button, then "a" as shown below in the diagram. Then do the same thing for the next vector except store this as "b"
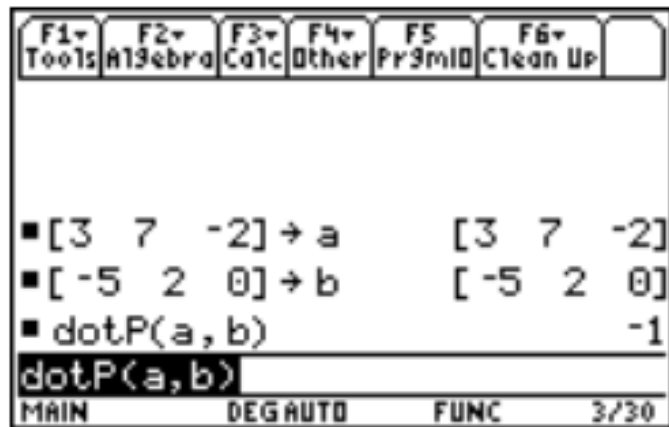
$$\vec{F} = 3\hat{i} + 7\hat{j} - 2\hat{k}$$

$$\vec{d} = -5\hat{i} + 2\hat{j}$$

Now that you have assigned each vector a variable. On the calculator go to CATALOG and find dotP(from the menu). Hit enter. Then type in a,b and close the parenthesis as shown in the diagrams.

Dot products using the TI-89

$$\vec{F} = 3\hat{i} + 7\hat{j} - 2\hat{k}$$

$$\vec{d} = -5\hat{i} + 2\hat{j}$$

We see we get "-1" for the WORK. What does this mean?

Remember, the dot product gives you a SCALAR which has NO DIRECTION. The negative here simply means that the Force and Displacement oppose each other yet are still PARALLEL.

An example would be the WORK done by friction to slow a base runner down.

Displacement

Force of Friction

The "Cross" Product (Vector Multiplication)

Multiplying 2 vectors sometimes gives you a VECTOR quantity which we call the **VECTOR CROSS PRODUCT**.
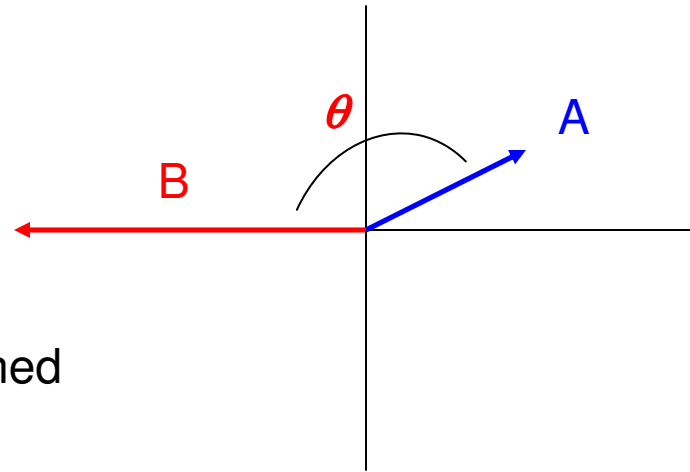
In polar notation consider 2 vectors:
A = |A| < θ$_1$ &   B = |B| < θ$_2$

The cross product between A and B produces a VECTOR quantity. The magnitude of the vector product is defined as:

$$\left| AxB \right| = \left| A \right| \left| B \right| \sin \phi$$

Where θ is the NET angle between the two vectors. As shown in the figure.

The Vector Cross Product

Example:
Let A = |5| < 30, Let B = |12| < 180
What is A "cross" B?
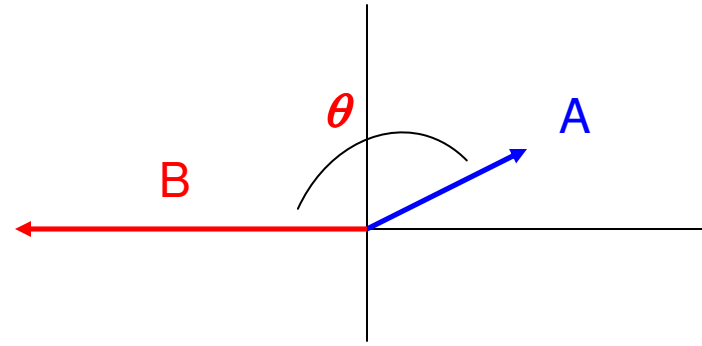
$$A \otimes B = |A||B| \sin \theta = |12||5| \sin 150$$

$$A \otimes B = 30\hat{k}$$

What about the direction???? Positive k-hat???
We can use what is called the RIGHT HAND THUMB RULE.
• Fingers are the first vector, A
• Palm is the second vector, B
• **Thumb is the direction of the cross product.**
• Cross your fingers, A, towards, B so that they CURL. The direction it moves will be either clockwise (NEGATIVE) or counter clockwise (POSITIVE)
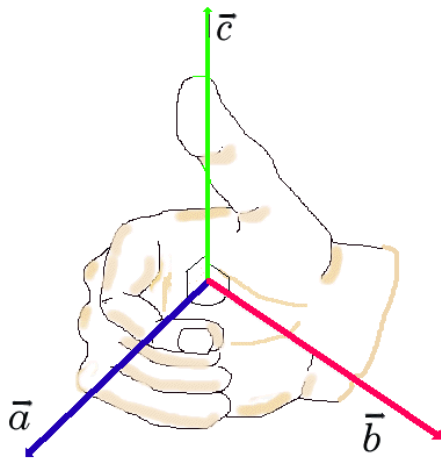
In our example, the thumb points OUTWARD which is the Z axis and thus our answer would be 30 k-hat since the curl moves counter clockwise.
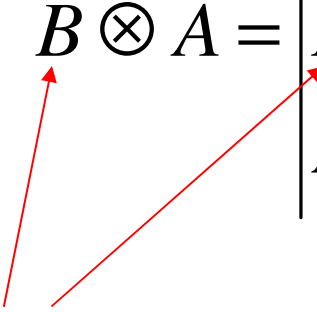
Cross Products and Unit Vectors

Consider:

$$A = A_x i + A_y j + A_z k$$
$$B = B_x i + B_y j + B_z k$$

The cross product between B and A produces a VECTOR of which a 3x3 matrix is need to evaluate the magnitude and direction.

You start by making a 3x3 matrix with 3 columns, one for i, j, & k-hat. The components then go under each appropriate column.

$$B \otimes A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

Since B is the first vector it comes first in the matrix

Cross Products and Unit Vectors

$$B \otimes A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

You then make an X in the columns OTHER THAN the unit vectors you are working with.
- **For "i" , cross j x k**
- **For "j" , cross i x k**
- **For "k" , cross i x j**

Let's start with the i-hat vector: We cross j x k

$$\hat{i} = (B_y A_z) - (B_z A_y)$$

Now the j-hat vector: We cross i x k

$$\hat{j} = (B_z A_x) - (B_x A_z)$$

Now the k-hat vector: We cross i x j

$$\hat{k} = (B_x A_y) - (B_y A_x)$$

Example

Example: Let $A = 3i - 4j + 5k$ and $B = -2i - 4j - 6k$. What is $B \times A$?

$$B \times A = \begin{vmatrix} i & j & k \\ -2 & -4 & -6 \\ 3 & -4 & 5 \end{vmatrix}$$

Let's start with the i-hat vector: We cross j x k

$$\hat{i} = (-4)(5) - (-6)(-4) = -44$$

Now the j-hat vector: We cross i x k

$$\hat{j} = (-6)(3) - (-2)(5) = -8$$

Now the k-hat vector: We cross i x j
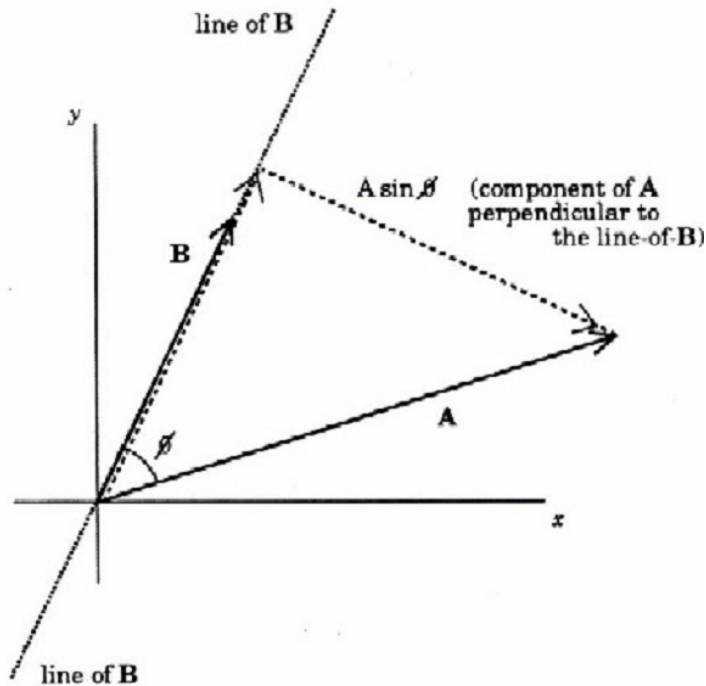
$$\hat{k} = (-2)(-4) - (-4)(3) = 20$$

The final answer would be:

$$B \otimes A = -44\hat{i} - 8\hat{j} + 20\hat{k}$$

The significance of the cross product

$$\left|AxB\right| = \left|A\right|\left|B\right|\sin\phi$$

In this figure, vector A has been split into 2 components, one PARALLEL to vector B and one PERPENDICULAR to vector B. Notice that the component perpendicular to vector B has a magnitude of |A|sin θ

THEREFORE when you find the CROSS PRODUCT, the result is:
  i) The MAGNITUDE of one vector, in this case |B| and,
  ii) The MAGNITUDE of the 2nd vector's component that runs
    perpendicular to the first vector. (that is where the sine comes from)

Cross Products in Physics

There are many cross products in physics. You will see the matrix system when you learn to analyze circuits with multiple batteries. The cross product system will also be used in mechanics (rotation) as well as understanding the behavior of particles in magnetic fields.
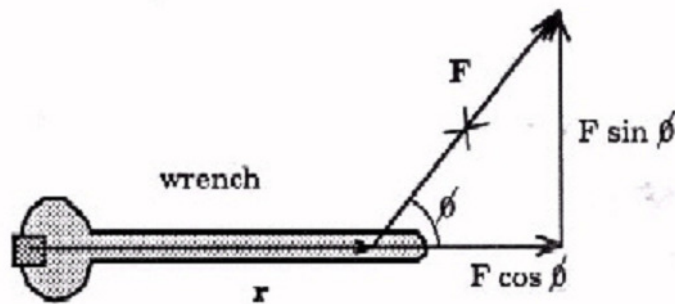
A force **F** is applied to a wrench a displacement **r** from a specific point of rotation (ie. a bolt).

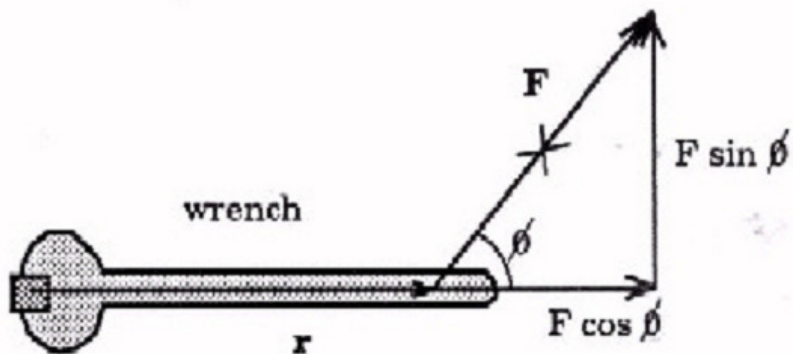Common sense will tell us the larger **r** is the easier it will be to turn the bolt.

But which part of **F** actually causes the wrench to turn?   *|F| Sin θ*

$$A \otimes B = |A||B|\sin\theta$$

$$\vec{F} \otimes \vec{r} = |F||r|\sin\theta$$

Cross Products in Physics

$$A \otimes B = |A||B|\sin\theta$$

$$\vec{F} \otimes \vec{r} = |F||r|\sin\theta$$

What about the DIRECTION?

Which way will the wrench turn?   **Counter Clockwise**

 Is the turning direction positive or negative?   **Positive**

Which way will the BOLT move? IN or OUT of the page?   **OUT**

You have to remember that cross products give you a direction on the OTHER axis from the 2 you are crossing. So if "r" is on the x-axis and "F" is on the y-axis, the cross products direction is on the z-axis. In this case, a POSITIVE k-hat.

Cross products using the TI-89

Let's clear the stored variables in the TI-89
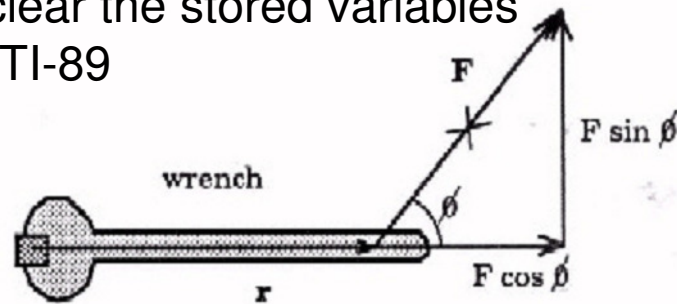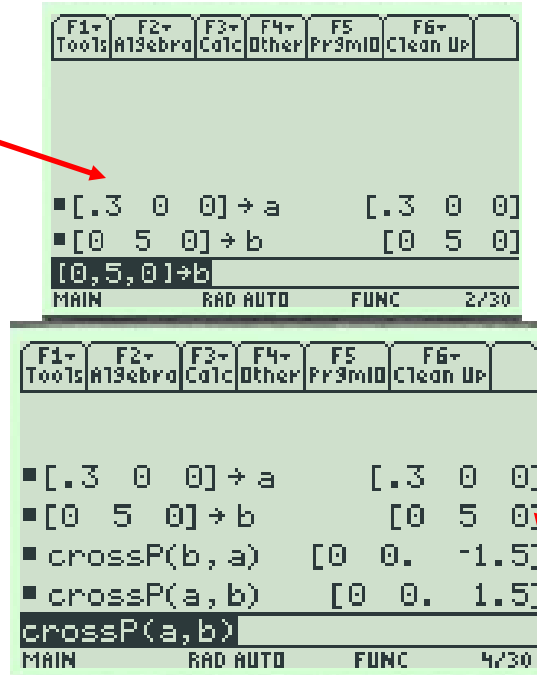
$$A = r = 0.30m$$

$$B = F = 5N$$

$$B \otimes A = ?$$

Since "r" is ONLY on the x-axis, it ONLY has an i-hat value. Enter, zeros for the other unit vectors.

Using **CATALOG**, find **crossP** from the menu.

Cross BxA, then AxB

Look at the answers carefully! Crossing BxA shows that we have a direction on the z-axis. Since it is negative it rotates CW. This mean the BOLT is being tightened or moves IN to the page. Crossing AxB, causes the bolt to loosen as it moves OUT of the page