Chapter 2

Limits and Continuity

An Economic Injury Level (EIL) is a measurement of the fewest number of insect pests that will cause economic damage to a crop or forest. It has been estimated that monitoring pest populations and establishing EILs can reduce pesticide use by 30%–50%.

Accurate population estimates are crucial for determining EILs. A population density of one insect pest can be approximated by

$$D(t) = \frac{t^2}{90} + \frac{t}{3}$$

pests per plant, where t is the number of days since initial infestation. What is the rate of change of this population density when the population density is equal to the EIL of 20 pests per plant? Section 2.4 can help answer this question.

Chapter 2 Overview

The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits is to test functions for continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior. They also have special mathematical properties, not otherwise guaranteed.

2.1 Rates of Change and Limits

What you'll learn about

- Average and Instantaneous Speed
- Definition of Limit
- Properties of Limits
- One-sided and Two-sided Limits
- Sandwich Theorem

. . . and why

Limits can be used to describe continuity, the derivative, and the integral: the ideas giving the foundation of calculus.

Average and Instantaneous Speed

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit time—kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average speed of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}.$$

Now try Exercise 1.

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall*.

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant $t = 2$.

SOLUTION

Solve Numerically We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h}. \qquad (1)$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of h *close* to 0. When we do, we see a clear pattern (Table 2.1 on the next page). As h approaches 0, the average speed approaches the limiting value 64 ft/sec.

continued

Table 2.1 Average Speeds over Short Time Intervals Starting at $t = 2$

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

Length of Time Interval, h (sec)	Average Speed for Interval $\Delta y/\Delta t$ (ft/sec)
1	80
0.1	65.6
0.01	64.16
0.001	64.016
0.0001	64.0016
0.00001	64.00016

Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

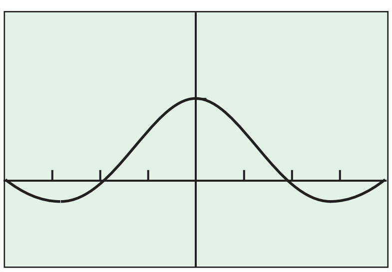
For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64$ ft/sec as h approaches 0. ***Now try Exercise 3.***

Definition of Limit

As in the preceding example, most limits of interest in the real world can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average speed was not defined at $h = 0$ but approached the limit 64 as h approached 0. We were able to see this numerically and to confirm it algebraically by eliminating h from the denominator. But we cannot always do that. For instance, we can see both graphically and numerically (Figure 2.1) that the values of $f(x) = (\sin x)/x$ approach 1 as x approaches 0.

We cannot eliminate the x from the denominator of $(\sin x)/x$ to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 75.



$[-2\pi, 2\pi]$ by $[-1, 2]$

(a)

X	Y1
−.3	.98507
−.2	.99335
−.1	.99833
0	ERROR
.1	.99833
.2	.99335
.3	.98507

Y1 = sin(X)/X

(b)

Figure 2.1 (a) A graph and (b) table of values for $f(x) = (\sin x)/x$ that suggest the limit of f as x approaches 0 is 1.

DEFINITION Limit

Assume f is defined in a neighborhood of c and let c and L be real numbers. The function ***f* has limit *L* as *x* approaches *c*** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \to c} f(x) = L.$$

The sentence $\lim_{x\to c} f(x) = L$ is read, "The limit of f of x as x approaches c equals L." The notation means that the values $f(x)$ of the function f approach or equal L as the values of x approach *(but do not equal)* c. Appendix A3 provides practice applying the definition of limit.

We saw in Example 2 that $\lim_{h\to 0}(64 + 16h) = 64$.

As suggested in Figure 2.1,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Figure 2.2 illustrates the fact that the existence of a limit as $x \to c$ never depends on how the function may or may not be defined at c. The function f has limit 2 as $x \to 1$ even though f is not defined at 1. The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.

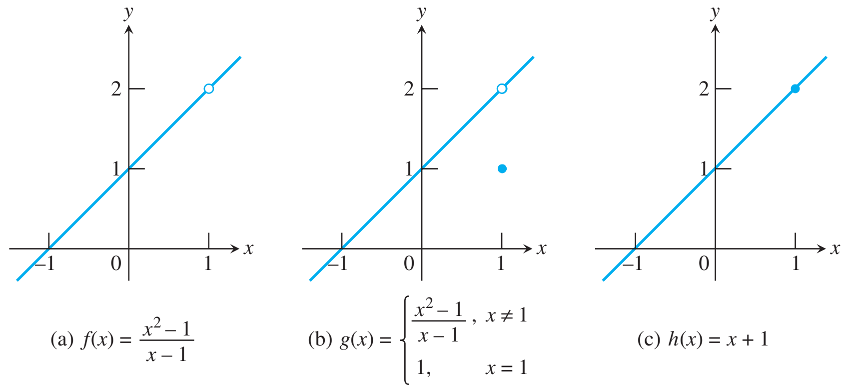


Figure 2.2 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$

Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

$$\lim_{x \to c} (k) = k \qquad \text{Limit of the function with constant value } k$$

and

$$\lim_{x \to c} (x) = c, \qquad \text{Limit of the identity function at } x = c$$

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$

   The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \; M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

continued

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

EXAMPLE 3 Using Properties of Limits

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, and the properties of limits to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

SOLUTION

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Product and Constant Multiple Rules

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$ Quotient Rule

$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Product Rule

Now try Exercises 5 and 6.

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, we can immediately work our way to limits of polynomial functions and most rational functions using substitution.

THEOREM 2 Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

EXAMPLE 4 Using Theorem 2

(a) $\lim_{x \to 3} [x^2(2 - x)] = (3)^2(2 - 3) = -9$

(b) $\lim_{x \to 2} \dfrac{x^2 + 2x + 4}{x + 2} = \dfrac{(2)^2 + 2(2) + 4}{2 + 2} = \dfrac{12}{4} = 3$

Now try Exercises 9 and 11.

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.

EXAMPLE 5 Using the Product Rule

Determine $\lim_{x \to 0} \dfrac{\tan x}{x}$.

SOLUTION

Solve Graphically The graph of $f(x) = (\tan x)/x$ in Figure 2.3 suggests that the limit exists and is about 1.

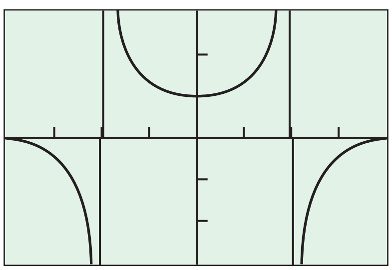
Confirm Analytically Using the analytic result of Exercise 75, we have

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) \qquad \tan x = \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} \qquad \text{Product Rule}$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1.$$

Now try Exercise 27.



$[-\pi, \pi]$ by $[-3, 3]$

Figure 2.3 The graph of $f(x) = (\tan x)/x$ suggests that $f(x) \to 1$ as $x \to 0$. (Example 5)

Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

EXAMPLE 6 Exploring a Nonexistent Limit

Use a graph to show that

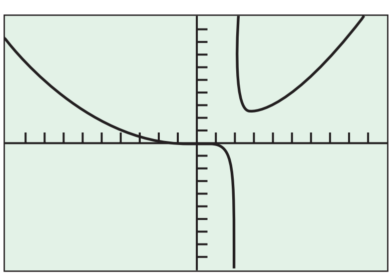
$$\lim_{x \to 2} \frac{x^3 - 1}{x - 2}$$

does not exist.

SOLUTION

Notice that the denominator is 0 when x is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 2.4 of $f(x) = (x^3 - 1)/(x - 2)$ strongly suggests that as $x \to 2$ from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

Now try Exercise 29.



$[-10, 10]$ by $[-100, 100]$

Figure 2.4 The graph of $f(x) = (x^3 - 1)/(x - 2)$ obtained using parametric graphing to produce a more accurate graph. (Example 6)

One-sided and Two-sided Limits

Sometimes the values of a function f tend to different limits as x approaches a number c from opposite sides. When this happens, we call the limit of f as x approaches c from the

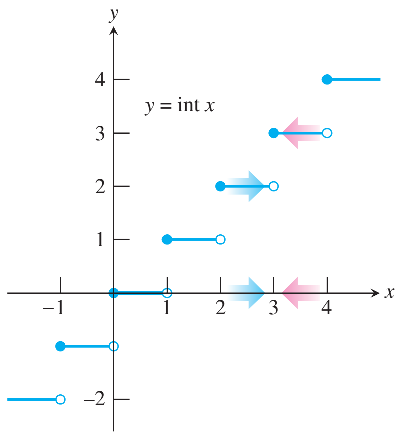


Figure 2.5 At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

right the **right-hand limit** of f at c and the limit as x approaches c from the left the **left-hand limit** of f at c. Here is the notation we use:

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x \to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int } x = 2.$$

The limit of int x as x approaches an integer n from the right is n, while the limit as x approaches n from the left is $n - 1$.

Now try Exercises 31 and 32.

We sometimes call $\lim_{x \to c} f(x)$ the **two-sided limit** of f at c to distinguish it from the *one-sided* right-hand and left-hand limits of f at c. Theorem 3 shows how these limits are related.

On the Far Side

If f is not defined to the left of $x = c$, then f does not have a left-hand limit at c. Similarly, if f is not defined to the right of $x = c$, then f does not have a right-hand limit at c.

THEOREM 3 One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 2.6 are true.

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

f has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x \to 1} f(x)$ does not exist.)

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2 = f(3) = \lim_{x \to 3} f(x)$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$.

At noninteger values of c between 0 and 4, f has a limit as $x \to c$.

Now try Exercise 37.

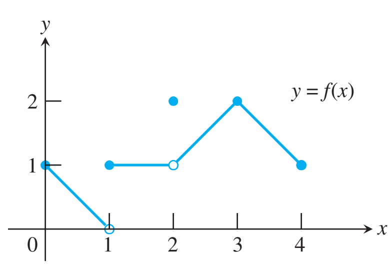


Figure 2.6 The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 < x \le 4. \end{cases}$$

(Example 8)

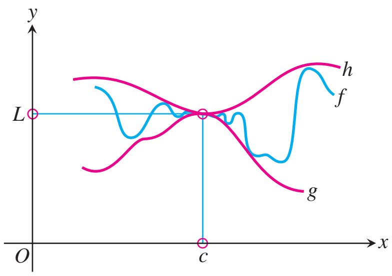


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

Sandwich Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function f whose values are sandwiched between the values of two other functions, g and h. If g and h have the same limit as $x \to c$, then f has that limit too, as suggested by Figure 2.7.

THEOREM 4 The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

EXAMPLE 9 Using the Sandwich Theorem

Show that $\lim_{x \to 0} [x^2 \sin(1/x)] = 0$.

SOLUTION

We know that the values of the sine function lie between –1 and 1. So, it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \le |x^2| \cdot 1 = x^2$$

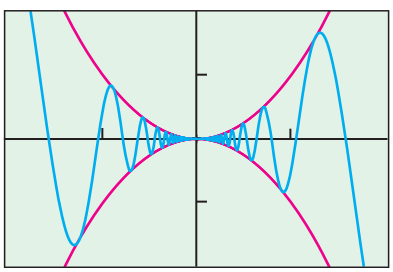
and

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Sandwich Theorem gives

$$\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 2.8 support this result.



[–0.2, 0.2] by [–0.02, 0.02]

Figure 2.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin(1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)

Quick Review 2.1 *(For help, go to Section 1.2.)*

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$ 0

2. $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$ $\dfrac{11}{12}$

3. $f(x) = \sin\left(\pi \dfrac{x}{2}\right)$ 0

4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \ge 2 \end{cases}$ $\dfrac{1}{3}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

5. $|x| < 4$ $-4 < x < 4$

6. $|x| < c^2$ $-c^2 < x < c^2$

7. $|x - 2| < 3$ $-1 < x < 5$

8. $|x - c| < d^2$ $c - d^2 < x < c + d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\dfrac{x^2 - 3x - 18}{x + 3}$ $x - 6$

10. $\dfrac{2x^2 - x}{2x^2 + x - 1}$ $\dfrac{x}{x + 1}$

Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall. 48 ft/sec
2. Find the average speed during the first 4 seconds of fall. 64 ft/sec
3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically. 96 ft/sec
4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically. 128 ft/sec

In Exercises 5 and 6, use $\lim_{x\to c} k = k$, $\lim_{x\to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x\to c} (2x^3 - 3x^2 + x - 1)$ $2c^3 - 3c^2 + c - 1$

6. $\lim_{x\to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$ $\dfrac{c^4 - c^3 + 1}{c^2 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x\to -1/2} 3x^2(2x - 1)$ $-\dfrac{3}{2}$

8. $\lim_{x\to -4} (x + 3)^{1998}$ 1

9. $\lim_{x\to 1} (x^3 + 3x^2 - 2x - 17)$ -15

10. $\lim_{y\to 2} \dfrac{y^2 + 5y + 6}{y + 2}$ 5

11. $\lim_{y\to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$ 0

12. $\lim_{x\to 1/2} \text{int } x$ 0

13. $\lim_{x\to -2} (x - 6)^{2/3}$ 4

14. $\lim_{x\to 2} \sqrt{x + 3}$ $\sqrt{5}$

In Exercises 15–18, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

15. $\lim_{x\to -2} \sqrt{x - 2}$ Expression not defined at $x = -2$. There is no limit.

16. $\lim_{x\to 0} \dfrac{1}{x^2}$ Expression not defined at $x = 0$. There is no limit.

17. $\lim_{x\to 0} \dfrac{|x|}{x}$ Expression not defined at $x = 0$. There is no limit.

18. $\lim_{x\to 0} \dfrac{(4 + x)^2 - 16}{x}$ Expression not defined at $x = 0$. Limit = 8.

In Exercises 19–28, determine the limit graphically. Confirm algebraically.

19. $\lim_{x\to 1} \dfrac{x - 1}{x^2 - 1}$ $\dfrac{1}{2}$

20. $\lim_{t\to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$ $\dfrac{1}{4}$

21. $\lim_{x\to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$ $-\dfrac{1}{2}$

22. $\lim_{x\to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$ $-\dfrac{1}{4}$

23. $\lim_{x\to 0} \dfrac{(2 + x)^3 - 8}{x}$ 12

24. $\lim_{x\to 0} \dfrac{\sin 2x}{x}$ 2

25. $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$ -1

26. $\lim_{x\to 0} \dfrac{x + \sin x}{x}$ 2

27. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$ 0

28. $\lim_{x\to 0} \dfrac{3 \sin 4x}{\sin 3x}$ 4

In Exercises 29 and 30, use a graph to show that the limit does not exist.

29. $\lim_{x\to 1} \dfrac{x^2 - 4}{x - 1}$

30. $\lim_{x\to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 31–36, determine the limit.

31. $\lim_{x\to 0^+} \text{int } x$ 0

32. $\lim_{x\to 0^-} \text{int } x$ -1

33. $\lim_{x\to 0.01} \text{int } x$ 0

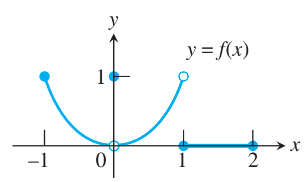
34. $\lim_{x\to 2^-} \text{int } x$ 1

35. $\lim_{x\to 0^+} \dfrac{x}{|x|}$ 1

36. $\lim_{x\to 0^-} \dfrac{x}{|x|}$ -1

In Exercises 37 and 38, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

37.



(a) $\lim_{x\to -1^+} f(x) = 1$ True

(b) $\lim_{x\to 0^-} f(x) = 0$ True

(c) $\lim_{x\to 0^-} f(x) = 1$ False

(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ True

(e) $\lim_{x\to 0} f(x)$ exists True

(f) $\lim_{x\to 0} f(x) = 0$ True

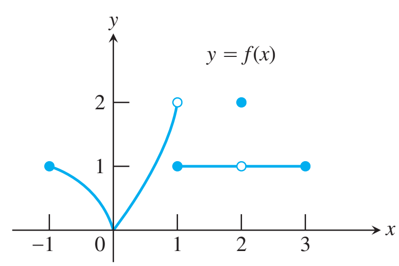
(g) $\lim_{x\to 0} f(x) = 1$ False

(h) $\lim_{x\to 1} f(x) = 1$ False

(i) $\lim_{x\to 1} f(x) = 0$ False

(j) $\lim_{x\to 2^-} f(x) = 2$ False

38.



(a) $\lim_{x\to -1^+} f(x) = 1$ True

(b) $\lim_{x\to 2} f(x)$ does not exist. False

(c) $\lim_{x\to 2} f(x) = 2$ False

(d) $\lim_{x\to 1^-} f(x) = 2$ True

(e) $\lim_{x\to 1^+} f(x) = 1$ True

(f) $\lim_{x\to 1} f(x)$ does not exist. True

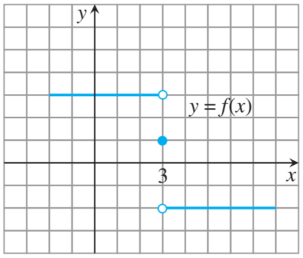
(g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ True

(h) $\lim_{x\to c} f(x)$ exists at every c in $(-1, 1)$. True

(i) $\lim_{x\to c} f(x)$ exists at every c in $(1, 3)$. True

29. Answers will vary. One possible graph is given by the window $[-4.7, 4.7]$ by $[-15, 15]$ with Xscl = 1 and Yscl = 5.
30. Answers will vary. One possible graph is given by the window $[-4.7, 4.7]$ by $[-15, 15]$ with Xscl = 1 and Yscl = 5.

In Exercises 39–44, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

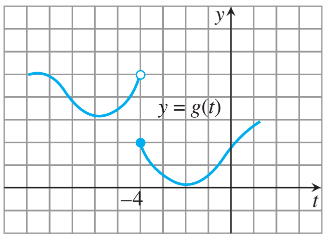
39. 

(a) $\lim_{x \to 3^-} f(x)$ 3

(b) $\lim_{x \to 3^+} f(x)$ −2

(c) $\lim_{x \to 3} f(x)$ No limit

(d) $f(3)$ 1

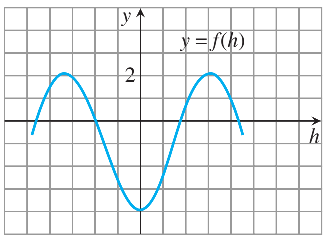
40. 

(a) $\lim_{t \to -4^-} g(t)$ 5

(b) $\lim_{t \to -4^+} g(t)$ 2

(c) $\lim_{t \to -4} g(t)$ No limit

(d) $g(-4)$ 2

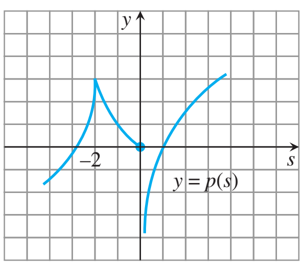
41. 

(a) $\lim_{h \to 0^-} f(h)$ −4

(b) $\lim_{h \to 0^+} f(h)$ −4

(c) $\lim_{h \to 0} f(h)$ −4

(d) $f(0)$ −4

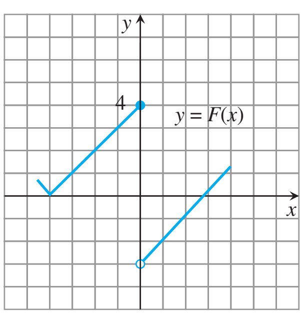
42. 

(a) $\lim_{s \to -2^-} p(s)$ 3

(b) $\lim_{s \to -2^+} p(s)$ 3

(c) $\lim_{s \to -2} p(s)$ 3

(d) $p(-2)$ 3

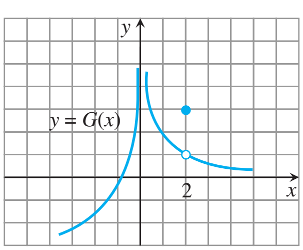
43. 

(a) $\lim_{x \to 0^-} F(x)$ 4

(b) $\lim_{x \to 0^+} F(x)$ −3

(c) $\lim_{x \to 0} F(x)$ No limit

(d) $F(0)$ 4

44. 

(a) $\lim_{x \to 2^-} G(x)$ 1

(b) $\lim_{x \to 2^+} G(x)$ 1

(c) $\lim_{x \to 2} G(x)$ 1

(d) $G(2)$ 3

In Exercises 45–48, match the function with the table.

45. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$ (c)

46. $y_1 = \dfrac{x^2 - x - 2}{x - 1}$ (b)

47. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$ (d)

48. $y_1 = \dfrac{x^2 + x - 2}{x + 1}$ (a)

X	Y1
.7	−.4765
.8	−.3111
.9	−.1526
1	0
1.1	.14762
1.2	.29091
1.3	.43043
X = .7	

(a)

X	Y1
.7	7.3667
.8	10.8
.9	20.9
1	ERROR
1.1	−18.9
1.2	−8.8
1.3	−5.367
X = .7	

(b)

X	Y1
.7	2.7
.8	2.8
.9	2.9
1	ERROR
1.1	3.1
1.2	3.2
1.3	3.3
X = .7	

(c)

X	Y1
.7	−.3
.8	−.2
.9	−.1
1	ERROR
1.1	.1
1.2	.2
1.3	.3
X = .7	

(d)

In Exercises 49 and 50, determine the limit.

49. Assume that $\lim_{x \to 4} f(x) = 0$ and $\lim_{x \to 4} g(x) = 3$.

(a) $\lim_{x \to 4} (g(x) + 3)$ 6 **(b)** $\lim_{x \to 4} x f(x)$ 0

(c) $\lim_{x \to 4} g^2(x)$ 9 **(d)** $\lim_{x \to 4} \dfrac{g(x)}{f(x) - 1}$ −3

50. Assume that $\lim_{x \to b} f(x) = 7$ and $\lim_{x \to b} g(x) = -3$.

(a) $\lim_{x \to b} (f(x) + g(x))$ 4 **(b)** $\lim_{x \to b} (f(x) \cdot g(x))$ −21

(c) $\lim_{x \to b} 4\,g(x)$ −12 **(d)** $\lim_{x \to b} \dfrac{f(x)}{g(x)}$ $-\dfrac{7}{3}$

In Exercises 51–54, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of *f*.

(b) Determine $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$.

(c) Writing to Learn Does $\lim_{x \to c} f(x)$ exist? If so, what is it? If not, explain.

51. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$ **(b)** Right-hand: 2 Left-hand: 1 **(c)** No, because the two one-sided limits are different.

52. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$ **(b)** Right-hand: 1 Left-hand: 1 **(c)** Yes. The limit is 1.

53. $c = 1, f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \geq 1 \end{cases}$ **(b)** Right-hand: 4 Left-hand: no limit **(c)** No, because the left-hand limit doesn't exist.

54. $c = -1, f(x) = \begin{cases} 1 - x^2, & x \neq -1 \\ 2, & x = -1 \end{cases}$ **(b)** Right-hand: 0 Left-hand: 0 **(c)** Yes. The limit is 0.

In Exercises 55–58, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

55. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$ **(b)** $(-2\pi, 0) \cup (0, 2\pi)$ **(c)** $c = 2\pi$ **(d)** $c = -2\pi$

56. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$ **(b)** $\left(-\pi, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ **(c)** $c = \pi$ **(d)** $c = -\pi$

57. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$ **(b)** $(0, 1) \cup (1, 2)$ **(c)** $c = 2$ **(d)** $c = 0$

58. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$ **(b)** $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **(c)** None **(d)** None

In Exercises 59–62, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

59. $\lim_{x\to 0} x \sin x$ 0

60. $\lim_{x\to 0} x^2 \sin x$ 0

61. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$ 0

62. $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$ 0

63. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall. 14.7 m/sec

(b) speed at the instant $t = 3$. 29.4 m/sec

64. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g. $g = \frac{5}{4}$

(b) Find the average speed for the fall. 5 m/sec

(c) With what speed did the rock hit the bottom? 10 m/sec

66. True.

$$\lim_{x\to 0}\left(\frac{x + \sin x}{x}\right) = \lim_{x\to 0}\left(1 + \frac{\sin x}{x}\right) = 1 + \lim_{x\to 0}\frac{\sin x}{x} = 2$$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

65. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer. True. Definition of limit.

66. True or False $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 67–70, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

67. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$? C

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

68. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$? B

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

69. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$? E

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

70. Multiple Choice What is the value of $f(1)$? C

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

Explorations

In Exercises 71–74, complete the following tables and state what you believe $\lim_{x\to 0} f(x)$ to be.

(a)

x	-0.1	-0.01	-0.001	-0.0001	…
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	…
$f(x)$	?	?	?	?	

71. $f(x) = x \sin \dfrac{1}{x}$

72. $f(x) = \sin \dfrac{1}{x}$

73. $f(x) = \dfrac{10^x - 1}{x}$

74. $f(x) = x \sin(\ln |x|)$

75. Group Activity To prove that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$. Because the right-hand limit at zero depends only on the values of the function for positive x-values near zero.

(b) Use the figure to show that

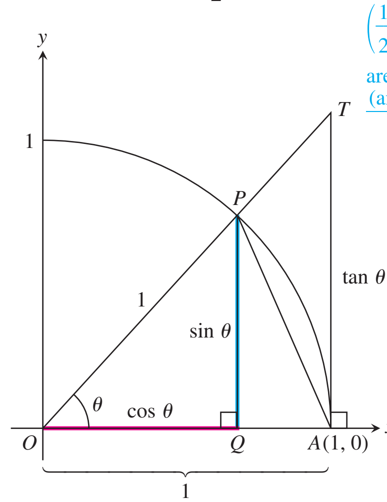
$$\text{area of } \triangle OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \triangle OAT = \frac{1}{2}\tan\theta.$$

Use: area of triangle $= \left(\frac{1}{2}\right)$(base)(height)

area of circular sector $= \dfrac{(\text{angle})(\text{radius})^2}{2}$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This is how the areas of the three regions compare.

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Multiply by 2 and divide by $\sin\theta$.

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Take reciprocals, remembering that all of the values involved are positive.

(f) Use the Sandwich Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

(i) Finally, show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

The two one-sided limits both exist and are equal to 1.

75. (f) The limits for $\cos\theta$ and 1 are both equal to 1. Since $\frac{\sin\theta}{\theta}$ is between them, it must also have a limit of 1.

(g) $\frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta}$

(h) If the function is symmetric about the y-axis, and the right-hand limit at zero is 1, then the left-hand limit at zero must also be 1.

Extending the Ideas

76. ***Controlling Outputs*** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x\to 2} f(x) = 2 = f(2)$. The limit can be found by substitution.

(b) Use a graph to estimate values for a and b so that $1.8 < f(x) < 2.2$ provided $a < x < b$. One possible answer: $a = 1.75$, $b = 2.28$

(c) Use a graph to estimate values for a and b so that $1.99 < f(x) < 2.01$ provided $a < x < b$. One possible answer: $a = 1.99$, $b = 2.01$

77. ***Controlling Outputs*** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$. $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

(b) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$. One possible answer: $a = 0.305$, $b = 0.775$

(c) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$. One possible answer: $a = 0.513$, $b = 0.535$

78. ***Limits and Geometry*** Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O be the origin and $(0, b)$ the y-intercept of the perpendicular bisector of line segment OP. Find $\lim_{P\to O} b$. $\frac{1}{2}$

2.2 Limits Involving Infinity

What you'll learn about

- Finite Limits as $x \to \pm\infty$
- Sandwich Theorem Revisited
- Infinite Limits as $x \to a$
- End Behavior Models
- "Seeing" Limits as $x \to \pm\infty$

. . . and why

Limits can be used to describe the behavior of functions for numbers large in absolute value.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when we say "the limit of f as x approaches infinity" we mean the limit of f as x moves increasingly far to the right on the number line. When we say "the limit of f as x approaches negative infinity ($-\infty$)" we mean the limit of f as x moves increasingly far to the left. (The limit in each case may or may not exist.)

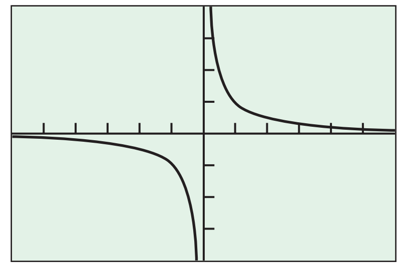
Looking at $f(x) = 1/x$ (Figure 2.9), we observe

(a) as $x \to \infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to\infty} (1/x) = 0,$$

(b) as $x \to -\infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to-\infty} (1/x) = 0.$$



[–6, 6] by [–4, 4]

Figure 2.9 The graph of $f(x) = 1/x$

We say that the line $y = 0$ is a *horizontal asymptote* of the graph of f.

DEFINITION Horizontal Asymptote

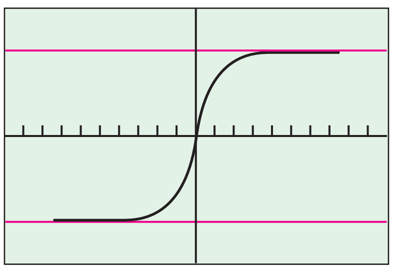
The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

The graph of $f(x) = 2 + (1/x)$ has the single horizontal asymptote $y = 2$ because

$$\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2 \quad \text{and} \quad \lim_{x\to-\infty}\left(2 + \frac{1}{x}\right) = 2.$$

A function can have more than one horizontal asymptote, as Example 1 demonstrates.



[–10, 10] by [–1.5, 1.5]

(a)

X	Y1	
0	0	
1	.7071	
2	.8944	
3	.9487	
4	.9701	
5	.9806	
6	.9864	

Y1 = X/√(X² + 1)

X	Y1	
-6	-.9864	
-5	-.9806	
-4	-.9701	
-3	-.9487	
-2	-.8944	
-1	-.7071	
0	0	

Y1 = X/√(X² + 1)

(b)

Figure 2.10 (a) The graph of $f(x) = x/\sqrt{x^2 + 1}$ has two horizontal asymptotes, $y = -1$ and $y = 1$. (b) Selected values of f. (Example 1)

EXAMPLE 1 Looking for Horizontal Asymptotes

Use graphs and tables to find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and identify all horizontal asymptotes of $f(x) = x/\sqrt{x^2 + 1}$.

SOLUTION

Solve Graphically Figure 2.10a shows the graph for $-10 \le x \le 10$. The graph climbs rapidly toward the line $y = 1$ as x moves away from the origin to the right. On our calculator screen, the graph soon becomes indistinguishable from the line. Thus $\lim_{x\to\infty} f(x) = 1$. Similarly, as x moves away from the origin to the left, the graph drops rapidly toward the line $y = -1$ and soon appears to overlap the line. Thus $\lim_{x\to-\infty} f(x) = -1$. The horizontal asymptotes are $y = 1$ and $y = -1$.

continued

Confirm Numerically The table in Figure 2.10b confirms the rapid approach of $f(x)$ toward 1 as $x \to \infty$. Since f is an odd function of x, we can expect its values to approach -1 in a similar way as $x \to -\infty$. ***Now try Exercise 5.***

Sandwich Theorem Revisited

The Sandwich Theorem also holds for limits as $x \to \pm\infty$.

EXAMPLE 2 Finding a Limit as x Approaches ∞

Find $\lim_{x\to\infty} f(x)$ for $f(x) = \frac{\sin x}{x}$.

SOLUTION

Solve Graphically and Numerically The graph and table of values in Figure 2.11 suggest that $y = 0$ is the horizontal asymptote of f.

Confirm Analytically We know that $-1 \le \sin x \le 1$. So, for $x > 0$ we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

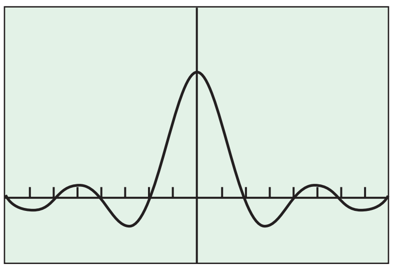
Therefore, by the Sandwich Theorem,

$$0 = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\frac{1}{x} = 0.$$

Since $(\sin x)/x$ is an even function of x, we can also conclude that

$$\lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

Now try Exercise 9.



[-4π, 4π] by [–0.5, 1.5]

(a)

X	Y1	
100	–.0051	
200	–.0044	
300	–.0033	
400	–.0021	
500	–9E–4	
600	7.4E–5	
700	7.8E–4	

Y1 = sin(X)/X

(b)

Figure 2.11 (a) The graph of $f(x) = (\sin x)/x$ oscillates about the x-axis. The amplitude of the oscillations decreases toward zero as $x \to \pm\infty$. (b) A table of values for f that suggests $f(x) \to 0$ as $x \to \infty$. (Example 2)

Limits at infinity have properties similar to those of finite limits.

THEOREM 5 Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = \frac{L}{M}, M \ne 0$
6. *Power Rule:* If r and s are integers, $s \ne 0$, then

$$\lim_{x\to\pm\infty}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

We can use Theorem 5 to find limits at infinity of functions with complicated expressions, as illustrated in Example 3.

EXAMPLE 3 Using Theorem 5

Find $\lim_{x\to\infty} \frac{5x + \sin x}{x}$.

SOLUTION

Notice that

$$\frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}.$$

So,

$$\lim_{x\to\infty} \frac{5x + \sin x}{x} = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{\sin x}{x} \qquad \text{Sum Rule}$$

$$= 5 + 0 = 5. \qquad \text{Known Values}$$

Now try Exercise 25.

EXPLORATION 1 Exploring Theorem 5

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let $f(x) = 5x + \sin x$ and $g(x) = x$. Do the limits as $x\to\infty$ of f and g exist? Can we apply the Quotient Rule to $\lim_{x\to\infty} f(x)/g(x)$? Explain. Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Describe the behavior of f and g as $x\to\infty$. Can we apply the Sum Rule to $\lim_{x\to\infty} (f(x) + g(x))$? Explain. Does the limit of the sum exist?
3. Let $f(x) = \ln(2x)$ and $g(x) = \ln(x + 1)$. Find the limits as $x\to\infty$ of f and g. Can we apply the Difference Rule to $\lim_{x\to\infty} (f(x) - g(x))$? Explain. Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying Theorem 5?

Infinite Limits as $x\to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, we say that $\lim_{x\to a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as $x\to a$, we say that $\lim_{x\to a} f(x) = -\infty$.

Looking at $f(x) = 1/x$ (Figure 2.9, page 70), we observe that

$$\lim_{x\to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x\to 0^-} 1/x = -\infty.$$

We say that the line $x = 0$ is a *vertical asymptote* of the graph of f.

DEFINITION Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

EXAMPLE 4 Finding Vertical Asymptotes

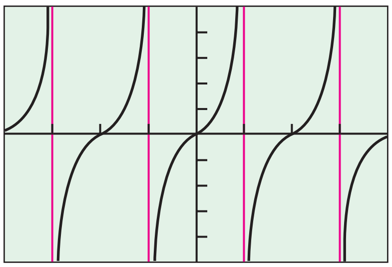
Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.

SOLUTION

The values of the function approach ∞ on either side of $x = 0$.

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

The line $x = 0$ is the only vertical asymptote. ***Now try Exercise 27.***



[-2π, 2π] by [–5, 5]

Figure 2.12 The graph of $f(x) = \tan x$ has a vertical asymptote at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. (Example 5)

We can also say that $\lim_{x\to 0} (1/x^2) = \infty$. We can make no such statement about $1/x$.

EXAMPLE 5 Finding Vertical Asymptotes

The graph of $f(x) = \tan x = (\sin x)/(\cos x)$ has infinitely many vertical asymptotes, one at each point where the cosine is zero. If a is an odd multiple of $\pi/2$, then

$$\lim_{x\to a^+} \tan x = -\infty \quad \text{and} \quad \lim_{x\to a^-} \tan x = \infty,$$

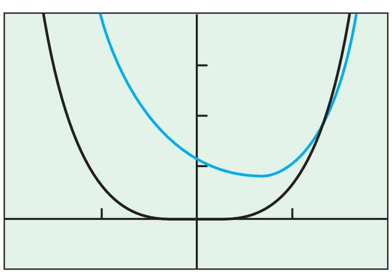
as suggested by Figure 2.12. ***Now try Exercise 31.***

You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that $\lim_{x\to 0} (\sin x)/x = 1$.

End Behavior Models

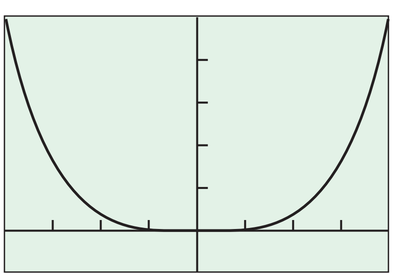
For numerically large values of x, we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

$y = 3x^4 - 2x^3 + 3x^2 - 5x + 6$



[–2, 2] by [–5, 20]

(a)



[–20, 20] by [–100000, 500000]

(b)

Figure 2.13 The graphs of f and g, (a) distinct for $|x|$ small, are (b) nearly identical for $|x|$ large. (Example 6)

EXAMPLE 6 Modeling Functions For $|x|$ Large

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that while f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ large.

SOLUTION

Solve Graphically The graphs of f and g (Figure 2.13a), quite different near the origin, are virtually identical on a larger scale (Figure 2.13b).

Confirm Analytically We can test the claim that g models f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\
&= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\
&= 1,
\end{aligned}$$

convincing evidence that f and g behave alike for $|x|$ large. ***Now try Exercise 39.***

DEFINITION End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1.$

(b) a **left end behavior model** for f if and only if $\lim_{x\to-\infty} \dfrac{f(x)}{g(x)} = 1.$

If one function provides both a left and right end behavior model, it is simply called an **end behavior model.** Thus, $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ (Example 6).

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$. Overall, the end behavior of all polynomials behave like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

EXAMPLE 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \dfrac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ **(b)** $g(x) = \dfrac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

SOLUTION

(a) Notice that $2x^5$ is an end behavior model for the numerator of f, and $3x^2$ is one for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for f.

(b) Similarly, $2x^3$ is an end behavior model for the numerator of g, and $5x^3$ is one for the denominator of g. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for g. *Now try Exercise 43.*

Notice in Example 7b that the end behavior model for g, $y = 2/5$, is also a horizontal asymptote of the graph of g, while in 7a, the graph of f does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function's right and left end behavior models need not be the same function.

EXAMPLE 8 Finding End Behavior Models

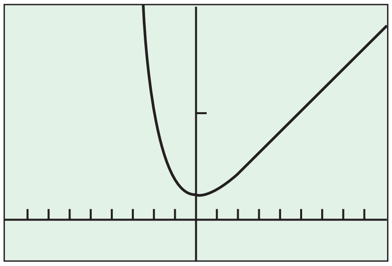
Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f while $h(x) = e^{-x}$ is a left end behavior model for f.

SOLUTION

On the right,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + e^{-x}}{x} = \lim_{x\to\infty} \left(1 + \frac{e^{-x}}{x}\right) = 1 \text{ because } \lim_{x\to\infty} \frac{e^{-x}}{x} = 0.$$

continued



[–9, 9] by [–2, 10]

Figure 2.14 The graph of $f(x) = x + e^{-x}$ looks like the graph of $g(x) = x$ to the right of the y-axis, and like the graph of $h(x) = e^{-x}$ to the left of the y-axis. (Example 8)

On the left,

$$\lim_{x\to-\infty} \frac{f(x)}{h(x)} = \lim_{x\to-\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x\to-\infty} \left(\frac{x}{e^{-x}} + 1\right) = 1 \text{ because } \lim_{x\to-\infty} \frac{x}{e^{-x}} = 0.$$

The graph of f in Figure 2.14 supports these end behavior conclusions.

Now try Exercise 45.

"Seeing" Limits as $x\to\pm\infty$

We can investigate the graph of $y = f(x)$ as $x\to\pm\infty$ by investigating the graph of $y = f(1/x)$ as $x\to 0$.

EXAMPLE 9 Using Substitution

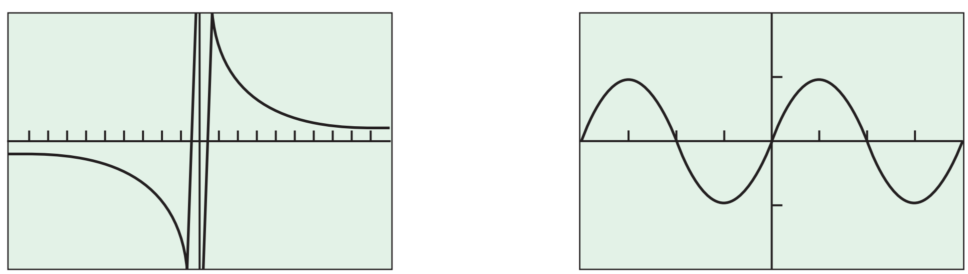
Find $\lim_{x\to\infty} \sin(1/x)$.

SOLUTION

Figure 2.15a suggests that the limit is 0. Indeed, replacing $\lim_{x\to\infty} \sin(1/x)$ by the equivalent $\lim_{x\to 0^+} \sin x = 0$ (Figure 2.15b), we find

$$\lim_{x\to\infty} \sin 1/x = \lim_{x\to 0^+} \sin x = 0.$$

Now try Exercise 49.



[–10, 10] by [–1, 1]

(a)

[-2π, 2π] by [–2, 2]

(b)

Figure 2.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)

5. $q(x) = \frac{2}{3}$
$r(x) = -3x^2 - \left(\frac{5}{3}\right)x + \frac{7}{3}$

6. $q(x) = 2x^2 + 2x + 1$
$r(x) = -x^2 - x - 2$

Quick Review 2.2 *(For help, go to Section 1.2 and 1.5.)*

In Exercises 1–4, find f^{-1} and graph f, f^{-1}, and $y = x$ in the same square viewing window.

1. $f(x) = 2x - 3$ $\quad f^{-1}(x) = \frac{x+3}{2}$

2. $f(x) = e^x$ $\quad f^{-1}(x) = \ln(x)$

3. $f(x) = \tan^{-1} x$ $\quad f^{-1}(x) = \tan(x),\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

4. $f(x) = \cot^{-1} x$ $\quad f^{-1}(x) = \cot(x),\ 0 < x < \pi$

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1,\quad g(x) = 3x^3 + 4x - 5$

6. $f(x) = 2x^5 - x^3 + x - 1,\quad g(x) = x^3 - x^2 + 1$

In Exercises 7–10, write a formula for **(a)** $f(-x)$ and **(b)** $f(1/x)$. Simplify where possible.

7. $f(x) = \cos x$ $\quad$ **(a)** $f(-x) = \cos x$ $\quad$ **(b)** $f\left(\frac{1}{x}\right) = \cos\left(\frac{1}{x}\right)$

8. $f(x) = e^{-x}$ $\quad$ **(a)** $f(-x) = e^x$ $\quad$ **(b)** $f\left(\frac{1}{x}\right) = e^{-1/x}$

9. $f(x) = \frac{\ln x}{x}$ $\quad$ **(a)** $f(-x) = -\frac{\ln(-x)}{x}$ $\quad$ **(b)** $f\left(\frac{1}{x}\right) = -x\ln x$

10. $f(x) = \left(x + \frac{1}{x}\right)\sin x$
(a) $f(-x) = \left(x + \frac{1}{x}\right)\sin x$ $\quad$ **(b)** $f\left(\frac{1}{x}\right) = \left(\frac{1}{x} + x\right)\sin\left(\frac{1}{x}\right)$

Section 2.2 Exercises

In Exercises 1–8, use graphs and tables to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$ **(c)** Identify all horizontal asymptotes.

1. $f(x) = \cos\left(\dfrac{1}{x}\right)$ **(a)** 1 **(b)** 1 **(c)** $y = 1$

2. $f(x) = \dfrac{\sin 2x}{x}$ **(a)** 0 **(b)** 0 **(c)** $y = 0$

3. $f(x) = \dfrac{e^{-x}}{x}$ **(a)** 0 **(b)** $-\infty$ **(c)** $y = 0$

4. $f(x) = \dfrac{3x^3 - x + 1}{x + 3}$ **(a)** ∞ **(b)** ∞ **(c)** None

5. $f(x) = \dfrac{3x + 1}{|x| + 2}$ **(a)** 3 **(b)** -3 **(c)** $y = 3, y = -3$

6. $f(x) = \dfrac{2x - 1}{|x| - 3}$ **(a)** 2 **(b)** -2 **(c)** $y = 2, y = -2$

7. $f(x) = \dfrac{x}{|x|}$ **(a)** 1 **(b)** -1 **(c)** $y = 1, y = -1$

8. $f(x) = \dfrac{|x|}{|x| + 1}$ **(a)** 1 **(b)** 1 **(c)** $y = 1$

In Exercises 9–12, find the limit and confirm your answer using the Sandwich Theorem.

9. $\lim_{x\to\infty} \dfrac{1 - \cos x}{x^2}$ 0

10. $\lim_{x\to-\infty} \dfrac{1 - \cos x}{x^2}$ 0

11. $\lim_{x\to-\infty} \dfrac{\sin x}{x}$ 0

12. $\lim_{x\to\infty} \dfrac{\sin (x^2)}{x}$ 0

In Exercises 13–20, use graphs and tables to find the limits.

13. $\lim_{x\to 2^+} \dfrac{1}{x - 2}$ ∞

14. $\lim_{x\to 2^-} \dfrac{x}{x - 2}$ $-\infty$

15. $\lim_{x\to -3^-} \dfrac{1}{x + 3}$ $-\infty$

16. $\lim_{x\to -3^+} \dfrac{x}{x + 3}$ $-\infty$

17. $\lim_{x\to 0^+} \dfrac{\text{int } x}{x}$ 0

18. $\lim_{x\to 0^-} \dfrac{\text{int } x}{x}$ ∞

19. $\lim_{x\to 0^+} \csc x$ ∞

20. $\lim_{x\to (\pi/2)^+} \sec x$ $-\infty$

In Exercises 21–26, find $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$.

21. $y = \left(2 - \dfrac{x}{x + 1}\right)\left(\dfrac{x^2}{5 + x^2}\right)$ Both are 1

22. $y = \left(\dfrac{2}{x} + 1\right)\left(\dfrac{5x^2 - 1}{x^2}\right)$ Both are 5

23. $y = \dfrac{\cos (1/x)}{1 + (1/x)}$ Both are 1

24. $y = \dfrac{2x + \sin x}{x}$ Both are 2

25. $y = \dfrac{\sin x}{2x^2 + x}$ Both are 0

26. $y = \dfrac{x \sin x + 2 \sin x}{2x^2}$ Both are 0

In Exercises 27–34, **(a)** find the vertical asymptotes of the graph of $f(x)$. **(b)** Describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

27. $f(x) = \dfrac{1}{x^2 - 4}$ **(a)** $x = -2, x = 2$

28. $f(x) = \dfrac{x^2 - 1}{2x + 4}$ **(a)** $x = -2$

29. $f(x) = \dfrac{x^2 - 2x}{x + 1}$ **(a)** $x = -1$

30. $f(x) = \dfrac{1 - x}{2x^2 - 5x - 3}$ **(a)** $x = -\dfrac{1}{2}, x = 3$

31. $f(x) = \cot x$ **(a)** $x = k\pi$, k any integer

32. $f(x) = \sec x$ **(a)** $x = \dfrac{\pi}{2} + n\pi$, n any integer

33. $f(x) = \dfrac{\tan x}{\sin x}$

34. $f(x) = \dfrac{\cot x}{\cos x}$

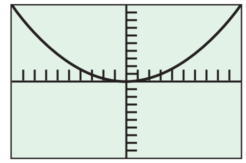
In Exercises 35–38, match the function with the graph of its end behavior model.

35. $y = \dfrac{2x^3 - 3x^2 + 1}{x + 3}$ **(a)**

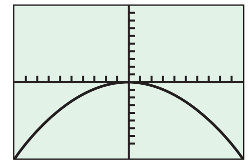
36. $y = \dfrac{x^5 - x^4 + x + 1}{2x^2 + x - 3}$ **(c)**

37. $y = \dfrac{2x^4 - x^3 + x^2 - 1}{2 - x}$ **(d)**

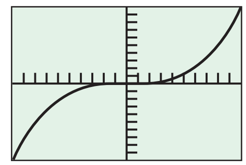
38. $y = \dfrac{x^4 - 3x^3 + x^2 - 1}{1 - x^2}$ **(b)**



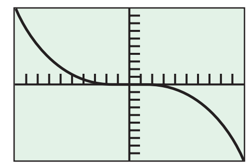
(a)



(b)



(c)



(d)

In Exercises 39–44, **(a)** find a power function end behavior model for f. **(b)** Identify any horizontal asymptotes.

39. $f(x) = 3x^2 - 2x + 1$ **(a)** $3x^2$ **(b)** None

40. $f(x) = -4x^3 + x^2 - 2x - 1$ **(a)** $-4x^3$ **(b)** None

41. $f(x) = \dfrac{x - 2}{2x^2 + 3x - 5}$ **(a)** $\dfrac{1}{2x}$ **(b)** $y = 0$

42. $f(x) = \dfrac{3x^2 - x + 5}{x^2 - 4}$ **(a)** 3 **(b)** $y = 3$

43. $f(x) = \dfrac{4x^3 - 2x + 1}{x - 2}$ **(a)** $4x^2$ **(b)** None

44. $f(x) = \dfrac{-x^4 + 2x^2 + x - 3}{x^2 - 4}$ **(a)** $-x^2$ **(b)** None

In Exercises 45–48, find **(a)** a simple basic function as a right end behavior model and **(b)** a simple basic function as a left end behavior model for the function.

45. $y = e^x - 2x$ **(a)** e^x **(b)** $-2x$

46. $y = x^2 + e^{-x}$ **(a)** x^2 **(b)** e^{-x}

47. $y = x + \ln |x|$ **(a)** x **(b)** x

48. $y = x^2 + \sin x$ **(a)** x^2 **(b)** x^2

In Exercises 49–52, use the graph of $y = f(1/x)$ to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

49. $f(x) = xe^x$ At ∞: ∞ At $-\infty$: 0

50. $f(x) = x^2e^{-x}$ At ∞: 0 At $-\infty$: ∞

51. $f(x) = \dfrac{\ln |x|}{x}$ At ∞: 0 At $-\infty$: 0

52. $f(x) = x \sin \dfrac{1}{x}$ At ∞: 1 At $-\infty$: 1

In Exercises 53 and 54, find the limit of $f(x)$ as **(a)** $x\to-\infty$, **(b)** $x\to\infty$, **(c)** $x\to 0^-$, and **(d)** $x\to 0^+$.

53. $f(x) = \begin{cases} 1/x, & x < 0 \\ -1, & x \ge 0 \end{cases}$ **(a)** 0 **(b)** -1 **(c)** $-\infty$ **(d)** -1

54. $f(x) = \begin{cases} \dfrac{x - 2}{x - 1}, & x \le 0 \\ 1/x^2, & x > 0 \end{cases}$ **(a)** 1 **(b)** 0 **(c)** 2 **(d)** ∞

Group Activity In Exercises 55 and 56, sketch a graph of a function $y = f(x)$ that satisfies the stated conditions. Include any asymptotes.

55. $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 5^-} f(x) = \infty$, $\lim_{x\to 5^+} f(x) = \infty$,
$\lim_{x\to\infty} f(x) = -1$, $\lim_{x\to -2^+} f(x) = -\infty$,
$\lim_{x\to -2^-} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

56. $\lim_{x\to 2} f(x) = -1$, $\lim_{x\to 4^+} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = \infty$,
$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 2$

57. $\frac{f_1(x)/f_2(x)}{g_1(x)/g_2(x)} = \frac{f_1(x)/g_1(x)}{f_2(x)/g_2(x)}$ As x goes to infinity, $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ both approach 1. Therefore, using the above equation, $\frac{f_1/f_2}{g_1/g_2}$ must also approach 1.

57. **Group Activity** ***End Behavior Models*** Suppose that $g_1(x)$ is a right end behavior model for $f_1(x)$ and that $g_2(x)$ is a right end behavior model for $f_2(x)$. Explain why this makes $g_1(x)/g_2(x)$ a right end behavior model for $f_1(x)/f_2(x)$.

58. **Writing to Learn** Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \infty$ or $-\infty$. Can $\lim_{x\to c} (f(x) + g(x))$ be determined? Explain.

59. True. For example, $f(x) = \frac{x}{\sqrt{x^2+1}}$ has $y = \pm 1$ as horizontal asymptotes.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

59. **True or False** It is possible for a function to have more than one horizontal asymptote. Justify your answer.

60. **True or False** If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = -\infty$. Justify your answer. False. Consider $f(x) = 1/x$.

61. **Multiple Choice** $\lim_{x\to 2^-} \frac{x}{x-2} =$ A

(A) $-\infty$ (B) ∞ (C) 1 (D) $-1/2$ (E) -1

62. **Multiple Choice** $\lim_{x\to 0} \frac{\cos(2x)}{x} =$ E

(A) 1/2 (B) 1 (C) 2 (D) cos 2 (E) does not exist

63. **Multiple Choice** $\lim_{x\to 0} \frac{\sin(3x)}{x} =$ C

(A) 1/3 (B) 1 (C) 3 (D) sin 3 (E) does not exist

64. **Multiple Choice** Which of the following is an end behavior for

$$f(x) = \frac{2x^3 - x^2 + x + 1}{x^3 - 1}?$$ D

(A) x^3 (B) $2x^3$ (C) $1/x^3$ (D) 2 (E) 1/2

Exploration

65. ***Exploring Properties of Limits*** Find the limits of f, g, and fg as $x\to c$.

(a) $f(x) = \frac{1}{x}$, $g(x) = x$, $c = 0$ $f\to -\infty$ as $x\to 0^-$, $f\to\infty$ as $x\to 0^+$, $g\to 0$, $fg\to 1$

(b) $f(x) = -\frac{2}{x^3}$, $g(x) = 4x^3$, $c = 0$ $f\to\infty$ as $x\to 0^-$, $f\to -\infty$ as $x\to 0^+$, $g\to 0$, $fg\to -8$

(c) $f(x) = \frac{3}{x-2}$, $g(x) = (x-2)^3$, $c = 2$ $f\to -\infty$ as $x\to 2^-$, $f\to\infty$ as $x\to 2^+$, $g\to 0$, $fg\to 0$

(d) $f(x) = \frac{5}{(3-x)^4}$, $g(x) = (x-3)^2$, $c = 3$ $x\to\infty$, $g\to 0$, $fg\to\infty$

(e) **Writing to Learn** Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Based on your observations in parts (a)–(d), what can you say about $\lim_{x\to c} (f(x)\cdot g(x))$? Nothing—you need more information to decide.

Extending the Ideas

66. ***The Greatest Integer Function***

(a) Show that $\frac{x-1}{x} < \frac{\text{int } x}{x} \le 1 \ (x > 0)$ and $\frac{x-1}{x} > \frac{\text{int } x}{x} \ge 1 \ (x < 0)$. This follows from $x - 1 < \text{int } x \le x$ which is true for all x. Dividing by x gives the result.

(b) Determine $\lim_{x\to\infty} \frac{\text{int } x}{x}$. 1

(c) Determine $\lim_{x\to -\infty} \frac{\text{int } x}{x}$. 1

67. ***Sandwich Theorem*** Use the Sandwich Theorem to confirm the limit as $x\to\infty$ found in Exercise 3.

68. **Writing to Learn** Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$. This is because as x approaches infinity, $\sin x$ continues to oscillate between 1 and -1 and doesn't approach any single real number.

In Exercises 69–71, find the limit. Give a convincing argument that the value is correct.

69. $\lim_{x\to\infty} \frac{\ln x^2}{\ln x}$ Limit = 2, because $\frac{\ln x^2}{\ln x} = \frac{2 \ln x}{\ln x} = 2$.

70. $\lim_{x\to\infty} \frac{\ln x}{\log x}$ Limit = ln (10), since $\frac{\ln x}{\log x} = \frac{\ln x}{\ln x / \ln 10} = \ln 10$.

71. $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x}$

Limit = 1. Since $\ln(x+1) = \ln x\left(1 + \frac{1}{x}\right) = \ln x + \ln\left(1 + \frac{1}{x}\right)$, $\frac{\ln(x+1)}{\ln x} = \frac{\ln x + \ln(1 + 1/x)}{\ln x} = 1 + \frac{\ln(1+1/x)}{\ln x}$. But as $x\to\infty$, $1 + \frac{1}{x}$ approaches 1, so $\ln\left(1 + \frac{1}{x}\right)$ approaches $\ln(1) = 0$. Also, as $x\to\infty$, $\ln x$ approaches infinity. This means the second term above approaches 0 and the limit is 1.

58. Yes. The limit of $(f + g)$ will be the same as the limit of g. This is because adding numbers that are very close to a given real number L will not have a significant effect on the value of $(f + g)$ since the values of g are becoming arbitrarily large.

Quick Quiz for AP* Preparation: Sections 2.1 and 2.2

You should solve the following problems without using a graphing calculator.

1. **Multiple Choice** Find $\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3}$, if it exists. D

(A) -1 (B) 1 (C) 2 (D) 5 (E) does not exist

2. **Multiple Choice** Find $\lim_{x\to 2^+} f(x)$, if it exists, where A

$$f(x) = \begin{cases} 3x + 1, & x < 2 \\ \frac{5}{x+1}, & x \ge 2 \end{cases}$$

(A) 5/3 (B) 13/3 (C) 7 (D) ∞ (E) does not exist

3. **Multiple Choice** Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}?$$ E

(A) $y = \frac{3}{2}x$ (B) $y = 0$ (C) $y = 2/3$ (D) $y = 7/5$ (E) $y = 3/2$

4. **Free Response** Let $f(x) = \frac{\cos x}{x}$.

(a) Find the domain and range of f. Domain: $(-\infty, 0) \cup (0, \infty)$; Range: $(-\infty, \infty)$.

(b) Is f even, odd, or neither? Justify your answer.

(c) Find $\lim_{x\to\infty} f(x)$. 0

(d) Use the Sandwich Theorem to justify your answer to part (c).

2.3 Continuity

What you'll learn about

- Continuity at a Point
- Continuous Functions
- Algebraic Combinations
- Composites
- Intermediate Value Theorem for Continuous Functions

. . . and why

Continuous functions are used to describe how a body moves through space and how the speed of a chemical reaction changes with time.

Continuity at a Point

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.16). In doing so, we are assuming that we are working with a *continuous function,* a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Any function $y = f(x)$ whose graph can be sketched in one continuous motion without lifting the pencil is an example of a continuous function.

Continuous functions are the functions we use to find a planet's closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies (Figure 2.17). As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

To understand continuity, we need to consider a function like the one in Figure 2.18, whose limits we investigated in Example 8, Section 2.1.

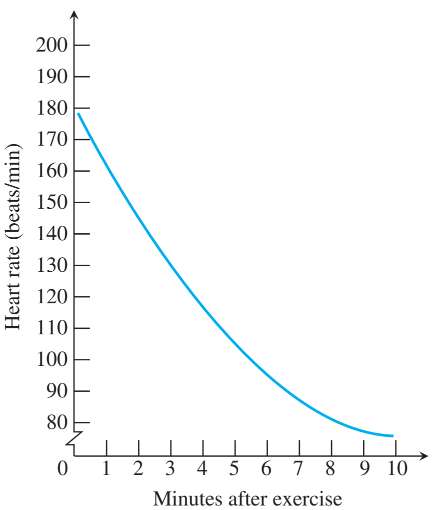


Figure 2.16 How the heartbeat returns to a normal rate after running.

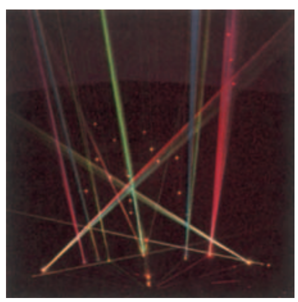


Figure 2.17 The laser was developed as a result of an understanding of the nature of the atom.

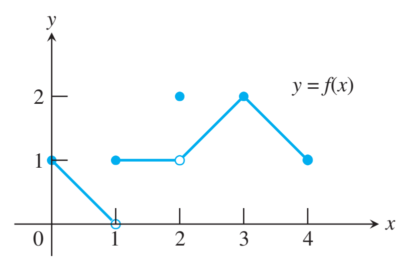


Figure 2.18 The function is continuous on [0, 4] except at $x = 1$ and $x = 2$. (Example 1)

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.18 is continuous, and the points at which f is discontinuous.

SOLUTION

The function f is continuous at every point in its domain [0, 4] except at $x = 1$ and $x = 2$. At these points there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\quad \lim_{x\to 0^+} f(x) = f(0).$

At $x = 4$, $\quad \lim_{x\to 4^-} f(x) = f(4).$

At $0 < c < 4,\ c \neq 1, 2,$ $\quad \lim_{x\to c} f(x) = f(c).$

continued

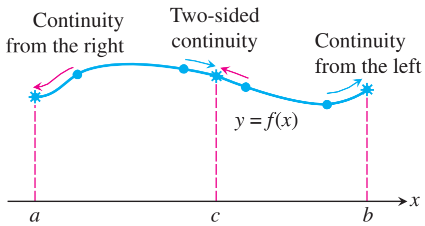


Figure 2.19 Continuity at points a, b, and c for a function $y = f(x)$ that is continuous on the interval $[a, b]$.

Points at which f is discontinuous:

At $x = 1$, $\lim_{x\to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x\to 2} f(x) = 1$, but $1 \neq f(2)$.

At $c < 0$, $c > 4$, these points are not in the domain of f.

Now try Exercise 5.

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit). (Figure 2.19)

DEFINITION Continuity at a Point

Interior Point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

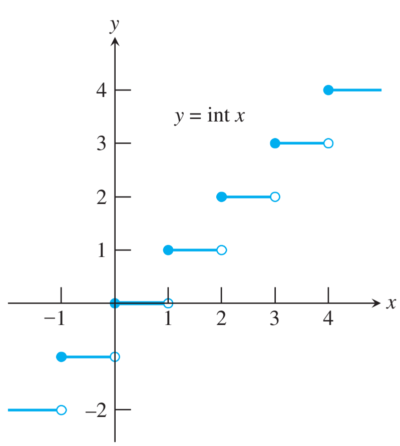


Figure 2.20 The function int x is continuous at every noninteger point. (Example 2)

EXAMPLE 2 Finding Points of Continuity and Discontinuity

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

SOLUTION

For the function to be continuous at $x = c$, the limit as $x \to c$ must exist and must equal the value of the function at $x = c$. The greatest integer function is discontinuous at every integer. For example,

$$\lim_{x\to 3^-} \text{int } x = 2 \quad \text{and} \quad \lim_{x\to 3^+} \text{int } x = 3$$

so the limit as $x \to 3$ does not exist. Notice that int $3 = 3$. In general, if n is any integer,

$$\lim_{x\to n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x\to n^+} \text{int } x = n,$$

so the limit as $x \to n$ does not exist.

The greatest integer function is continuous at every other real number. For example,

$$\lim_{x\to 1.5} \text{int } x = 1 = \text{int } 1.5.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x\to c} \text{int } x = n - 1 = \text{int } c.$$

Now try Exercise 7.

Shirley Ann Jackson
(1946–)

Distinguished scientist, Shirley Jackson credits her interest in science to her parents and excellent mathematics and science teachers in high school. She studied physics, and in 1973, became the first African American woman to earn a Ph.D. at the Massachusetts Institute of Technology. Since then, Dr. Jackson has done research on topics relating to theoretical material sciences, has received numerous scholarships and honors, and has published more than one hundred scientific articles.

Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: $\lim_{x \to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** The function in (f) has an **oscillating discontinuity:** it oscillates and has no limit as $x \to 0$.

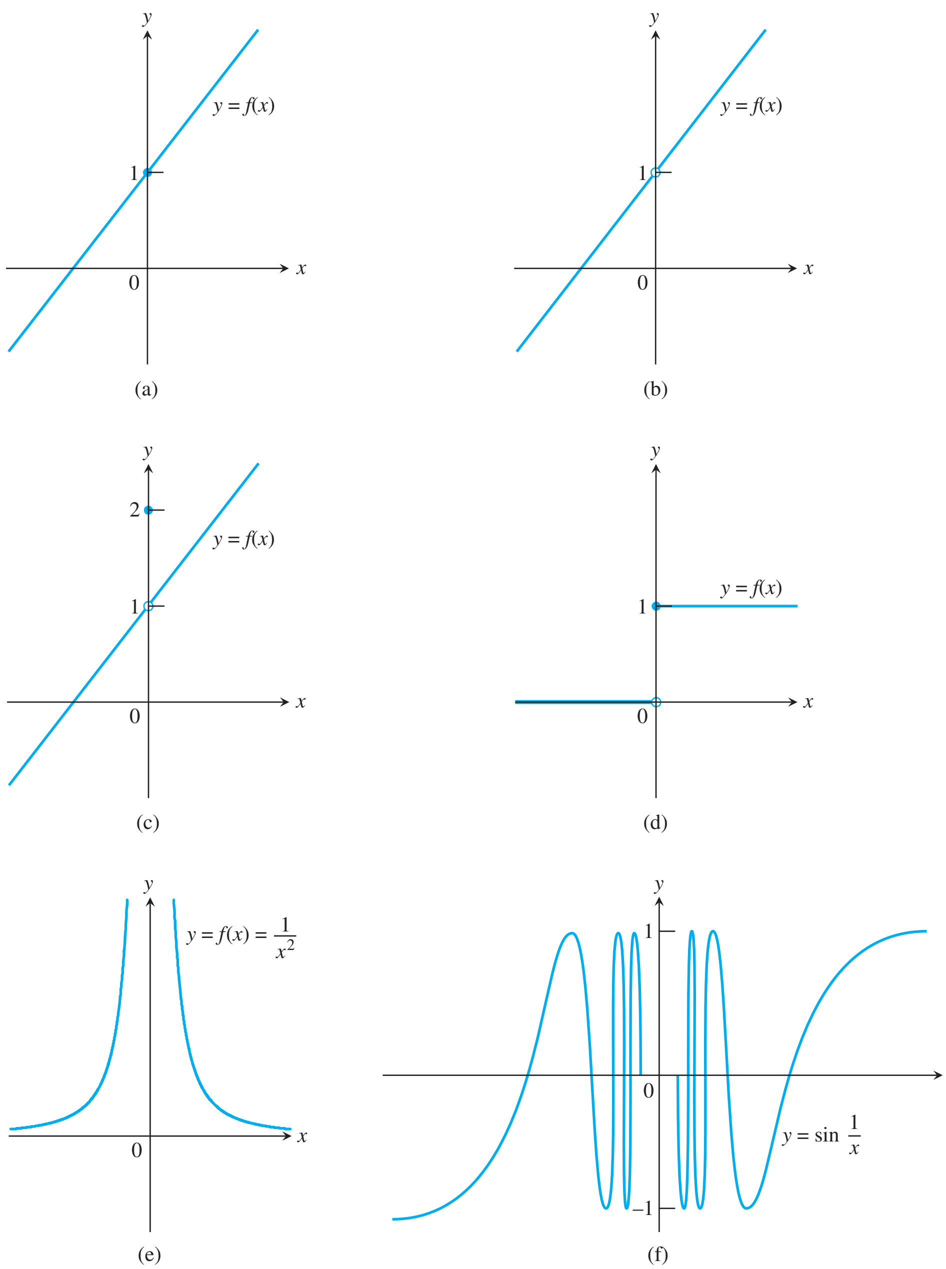


Figure 2.21 The function in part (a) is continuous at $x = 0$. The functions in parts (b)–(f) are not.

EXPLORATION 1 Removing a Discontinuity

Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$.

1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use zoom-in and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f.
5. Show that the *extended function*

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

Now try Exercise 25.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

EXAMPLE 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

Now try Exercise 31.

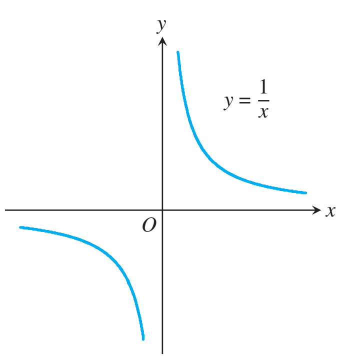


Figure 2.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

Polynomial functions f are continuous at every real number c because $\lim_{x \to c} f(x) = f(c)$. Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function $y = |x|$ is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like $y = \sqrt[n]{x}$ (n a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions.

Algebraic Combinations

As you may have guessed, algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.23). In this case, the limit as $x \to c$ is $g(f(c))$.

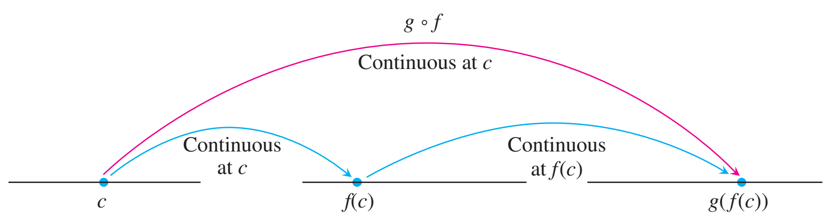


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

EXAMPLE 4 Using Theorem 7

Show that $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous.

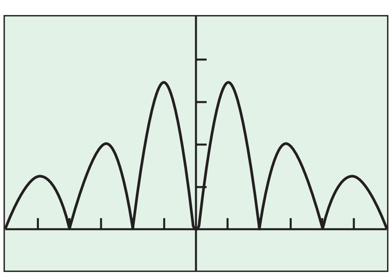
SOLUTION

The graph (Figure 2.24) of $y = |(x \sin x)/(x^2 + 2)|$ suggests that the function is continuous at every value of x. By letting

$$g(x) = |x| \quad \text{and} \quad f(x) = \frac{x \sin x}{x^2 + 2},$$

we see that y is the composite $g \circ f$.

We know that the absolute value function g is continuous. The function f is continuous by Theorem 6. Their composite is continuous by Theorem 7. ***Now try Exercise 33.***



$[-3\pi, 3\pi]$ by $[-0.1, 0.5]$

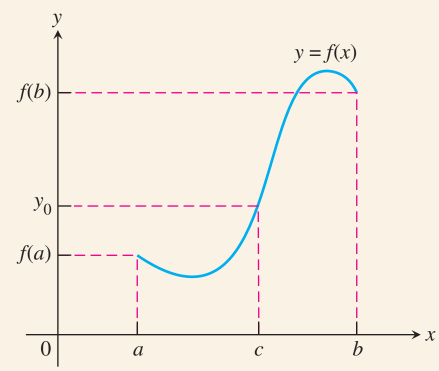
Figure 2.24 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous. (Example 4)

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *intermediate value property.* A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



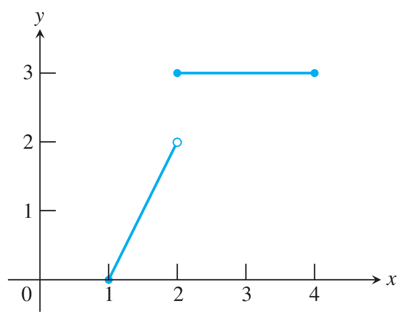


Figure 2.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected,** a single, unbroken curve, like the graph of sin x. It will not have jumps like those in the graph of the greatest integer function int x, or separate branches like we see in the graph of $1/x$.

Most graphers can plot points *(dot mode)*. Some can turn on pixels between plotted points to suggest an unbroken curve *(connected mode).* For functions, the connected format basically assumes that outputs *vary continuously* with inputs and do not jump from one value to another without taking on all values in between.

Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y =$ int (x) in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.

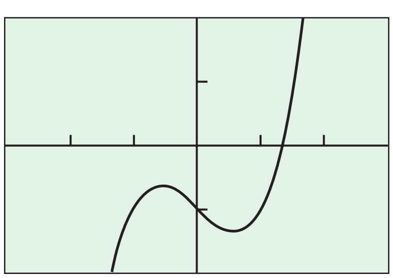
EXAMPLE 5 Using Theorem 8

Is any real number exactly 1 less than its cube?

SOLUTION

We answer this question by applying the Intermediate Value Theorem in the following way. Any such number must satisfy the equation $x = x^3 - 1$ or, equivalently, $x^3 - x - 1 = 0$. Hence, we are looking for a zero value of the continuous function $f(x) = x^3 - x - 1$ (Figure 2.26). The function changes sign between 1 and 2, so there must be a point c between 1 and 2 where $f(c) = 0$.

Now try Exercise 46.



[–3, 3] by [–2, 2]

Figure 2.26 The graph of $f(x) = x^3 - x - 1$. (Example 5)

6. $f(x) = \frac{1}{x^2} + 1, x > 0$ $(f \circ g)(x) = \frac{x}{x-1}, x > 1$

Quick Review 2.3 *(For help, go to Sections 1.2 and 2.1.)*

1. Find $\lim_{x \to -1} \frac{3x^2 - 2x + 1}{x^3 + 4}$. 2

2. Let $f(x) = \text{int } x$. Find each limit. **(a)** -2 **(b)** -1 **(c)** No limit **(d)** -1

(a) $\lim_{x \to -1^-} f(x)$ **(b)** $\lim_{x \to -1^+} f(x)$ **(c)** $\lim_{x \to -1} f(x)$ **(d)** $f(-1)$

3. Let $f(x) = \begin{cases} x^2 - 4x + 5, & x < 2 \\ 4 - x, & x \ge 2. \end{cases}$

Find each limit. **(a)** 1 **(b)** 2 **(c)** No limit **(d)** 2

(a) $\lim_{x \to 2^-} f(x)$ **(b)** $\lim_{x \to 2^+} f(x)$ **(c)** $\lim_{x \to 2} f(x)$ **(d)** $f(2)$

In Exercises 4–6, find the remaining functions in the list of functions: $f, g, f \circ g, g \circ f$.

4. $f(x) = \frac{2x - 1}{x + 5}$, $g(x) = \frac{1}{x} + 1$ $(f \circ g)(x) = \frac{x + 2}{6x + 1}, x \ne 0$ $(g \circ f)(x) = \frac{3x + 4}{2x - 1}, x \ne -5$

$g(x) = \sin x, x \ge 0$ $(f \circ g)(x) = \sin^2 x, x \ge 0$

5. $f(x) = x^2$, $(g \circ f)(x) = \sin x^2$, domain of $g = [0, \infty)$

6. $g(x) = \sqrt{x - 1}$, $(g \circ f)(x) = 1/x$, $x > 0$

7. Use factoring to solve $2x^2 + 9x - 5 = 0$. $x = \frac{1}{2}, -5$

8. Use graphing to solve $x^3 + 2x - 1 = 0$. $x \approx 0.453$

In Exercises 9 and 10, let

$$f(x) = \begin{cases} 5 - x, & x \le 3 \\ -x^2 + 6x - 8, & x > 3. \end{cases}$$

9. Solve the equation $f(x) = 4$. $x = 1$

10. Find a value of c for which the equation $f(x) = c$ has no solution. Any c in $[1, 2)$

Section 2.3 Exercises

5. All points not in the domain, i.e., all $x < -3/2$

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. $y = \frac{1}{(x + 2)^2}$ $x = -2$, infinite discontinuity

2. $y = \frac{x + 1}{x^2 - 4x + 3}$ $x = 1$ and $x = 3$, both infinite discontinuities

3. $y = \frac{1}{x^2 + 1}$ None

4. $y = |x - 1|$ None

5. $y = \sqrt{2x + 3}$

6. $y = \sqrt[3]{2x - 1}$ None

7. $y = |x|/x$ $x = 0$, jump discontinuity

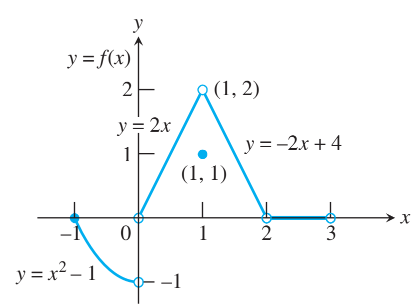
8. $y = \cot x$ $x = k\pi$ for all integers k, infinite discontinuity

9. $y = e^{1/x}$ $x = 0$, infinite discontinuity

10. $y = \ln(x + 1)$ All points not in the domain, i.e., all $x < -1$

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



11. (a) Does $f(-1)$ exist? Yes

(b) Does $\lim_{x \to -1^+} f(x)$ exist? Yes

(c) Does $\lim_{x \to -1^+} f(x) = f(-1)$? Yes

(d) Is f continuous at $x = -1$? Yes

12. (a) Does $f(1)$ exist? Yes

(b) Does $\lim_{x \to 1} f(x)$ exist? Yes

(c) Does $\lim_{x \to 1} f(x) = f(1)$? No

(d) Is f continuous at $x = 1$? No

13. (a) Is f defined at $x = 2$? (Look at the definition of f.) No

(b) Is f continuous at $x = 2$? No

14. At what values of x is f continuous? Everywhere in $[-1, 3)$ except for $x = 0, 1, 2$

15. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$? 0

16. What new value should be assigned to $f(1)$ to make the new function continuous at $x = 1$? 2

17. Writing to Learn Is it possible to extend f to be continuous at $x = 0$? If so, what value should the extended function have there? If not, why not? No, because the right-hand and left-hand limits are not the same at zero.

18. Writing to Learn Is it possible to extend f to be continuous at $x = 3$? If so, what value should the extended function have there? If not, why not? Yes. Assign the value 0 to $f(3)$.

In Exercises 19–24, **(a)** find each point of discontinuity. **(b)** Which of the discontinuities are removable? not removable? Give reasons for your answers.

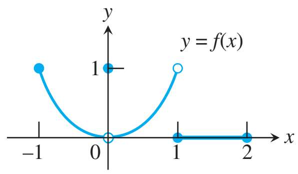
19. $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$ **(a)** $x = 2$ **(b)** Not removable, the one-sided limits are different.

20. $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$ **(a)** $x = 2$ **(b)** Removable, assign the value 1 to $f(2)$.

21. $f(x) = \begin{cases} \frac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$ **(a)** $x = 1$ **(b)** Not removable, it's an infinite discontinuity.

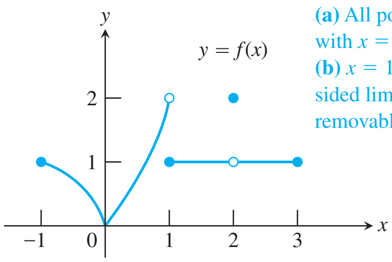
22. $f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$ **(a)** $x = -1$ **(b)** Removable, assign the value 0 to $f(-1)$.

23.



(a) All points not in the domain along with $x = 0, 1$
(b) $x = 0$ is a removable discontinuity, assign $f(0) = 0$. $x = 1$ is not removable, the two-sided limits are different.

24.



(a) All points not in the domain along with $x = 1, 2$
(b) $x = 1$ is not removable, the one-sided limits are different. $x = 2$ is a removable discontinuity, assign $f(2) = 1$.

In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. $f(x) = \dfrac{x^2 - 9}{x + 3}$, $x = -3$ $\quad y = x - 3$

26. $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$, $x = 1$ $\quad y = \dfrac{x^2 + x + 1}{x + 1}$

27. $f(x) = \dfrac{\sin x}{x}$, $x = 0$

28. $f(x) = \dfrac{\sin 4x}{x}$, $x = 0$

29. $f(x) = \dfrac{x - 4}{\sqrt{x} - 2}$, $x = 4$ $\quad y = \sqrt{x} + 2$

30. $f(x) = \dfrac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$, $x = 2$ $\quad y = \dfrac{x^2 - 2x - 15}{x + 2}$

In Exercises 31 and 32, explain why the given function is continuous.

31. $f(x) = \dfrac{1}{x - 3}$

32. $g(x) = \dfrac{1}{\sqrt{x - 1}}$

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. $f(x) = \sqrt{\left(\dfrac{x}{x + 1}\right)}$

34. $f(x) = \sin(x^2 + 1)$

35. $f(x) = \cos(\sqrt[3]{1 - x})$

36. $f(x) = \tan\left(\dfrac{x^2}{x^2 + 4}\right)$

Group Activity In Exercises 37–40, verify that the function is continuous and state its domain. Indicate which theorems you are using, and which functions you are assuming to be continuous.

37. $y = \dfrac{1}{\sqrt{x + 2}}$

38. $y = x^2 + \sqrt[3]{4 - x}$

39. $y = |x^2 - 4x|$

40. $y = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

In Exercises 41–44, sketch a possible graph for a function f that has the stated properties.

41. $f(3)$ exists but $\lim_{x \to 3} f(x)$ does not.

42. $f(-2)$ exists, $\lim_{x \to -2^+} f(x) = f(-2)$, but $\lim_{x \to -2} f(x)$ does not exist.

43. $f(4)$ exists, $\lim_{x \to 4} f(x)$ exists, but f is not continuous at $x = 4$.

44. $f(x)$ is continuous for all x except $x = 1$, where f has a nonremovable discontinuity.

$x \approx -0.724$ and $x \approx 1.221$

45. ***Solving Equations*** Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.

46. ***Solving Equations*** Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places. $x \approx -1.521$

47. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

is continuous. $\quad a = \dfrac{4}{3}$

48. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ ax + 1, & x > 2 \end{cases}$$

is continuous. $\quad a = 3$

49. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \geq -1 \end{cases}$$

is continuous. $\quad a = 4$

50. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \geq 1 \end{cases}$$

is continuous. $\quad a = -1$

51. **Writing to Learn** Explain why the equation $e^{-x} = x$ has at least one solution.

52. ***Salary Negotiation*** A welder's contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of \$36,500.

(a) Show that Luisa's salary is given by

$$y = 36{,}500(1.035)^{\text{int}\, t},$$

where t is the time, measured in years, since Luisa signed the contract.

(b) Graph Luisa's salary function. At what values of t is it continuous?

53. ***Airport Parking*** Valuepark charge \$1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is \$7.25. $\quad f(x) = \begin{cases} -1.10\ \text{int}(-x), & 0 \leq x \leq 6 \\ 7.25, & 6 < x \leq 24 \end{cases}$

(a) Write a formula that gives the charge for x hours with $0 \leq x \leq 24$. (*Hint:* See Exercise 52.)

(b) Graph the function in part (a). At what values of x is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

False. Consider $f(x) = 1/x$ which is continuous and has a point of discontinuity at $x = 0$.

54. **True or False** A continuous function cannot have a point of discontinuity. Justify your answer.

55. **True or False** It is possible to extend the definition of a function f at a jump discontinuity $x = a$ so that f is continuous at $x = a$. Justify your answer. True. If f has a jump discontinuity at $x = a$, then $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ so f is not continuous at $x = a$.

27. $y = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

28. $y = \begin{cases} \dfrac{\sin 4x}{x}, & x \neq 0 \\ 4, & x = 0 \end{cases}$

31. The domain of f is all real numbers $x \neq 3$. f is continuous at all those points so f is a continuous function.

32. The domain of g is all real numbers $x > 1$. f is continuous at all those points so g is a continuous function.

56. Multiple Choice On which of the following intervals is $f(x) = \dfrac{1}{\sqrt{x}}$ not continuous? B

(A) $(0, \infty)$ **(B)** $[0, \infty)$ **(C)** $(0, 2)$

(D) $(1, 2)$ **(E)** $[1, \infty)$

57. Multiple Choice Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x - 1}$? E

(A) $x = -1$ **(B)** $x = -1/2$ **(C)** $x = 0$

(D) $x = 1/2$ **(E)** $x = 1$

58. Multiple Choice Which of the following statements about the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$$

is not true? A

(A) $f(1)$ does not exist.

(B) $\lim_{x\to 0^+} f(x)$ exists.

(C) $\lim_{x\to 2^-} f(x)$ exists.

(D) $\lim_{x\to 1} f(x)$ exists.

(E) $\lim_{x\to 1} f(x) = f(1)$

59. Multiple Choice Which of the following points of discontinuity of

$$f(x) = \frac{x(x-1)(x-2)^2(x+1)^2(x-3)^2}{x(x-1)(x-2)(x+1)^2(x-3)^3}$$

is not removable? E

(A) $x = -1$ **(B)** $x = 0$ **(C)** $x = 1$

(D) $x = 2$ **(E)** $x = 3$

Exploration

60. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.

Domain of f: $(-\infty, -1) \cup (0, \infty)$

(a) Find the domain of f. **(b)** Draw the graph of f.

(c) Writing to Learn Explain why $x = -1$ and $x = 0$ are points of discontinuity of f. Because f is undefined there due to division by 0.

(d) Writing to Learn Are either of the discontinuities in part (c) removable? Explain. $x = 0$: removable, right-hand limit is 1; $x = -1$; not removable, infinite discontinuity

(e) Use graphs and tables to estimate $\lim_{x\to\infty} f(x)$. 2.718 or e

Extending the Ideas

61. *Continuity at a Point* Show that $f(x)$ is continuous at $x = a$ if and only if This is because $\lim_{h\to 0} f(a + h) = \lim_{x\to a} f(x)$.

$$\lim_{h\to 0} f(a + h) = f(a).$$

62. *Continuity on Closed Intervals* Let f be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

63. *Properties of Continuity* Prove that if f is continuous on an interval, then so is $|f|$.

64. *Everywhere Discontinuous* Give a convincing argument that the following function is not continuous at any real number.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

2.4 Rates of Change and Tangent Lines

What you'll learn about

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

. . . and why

The tangent line determines the direction of a body's motion at every point along its path.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval [1, 3].

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval [1, 3] is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

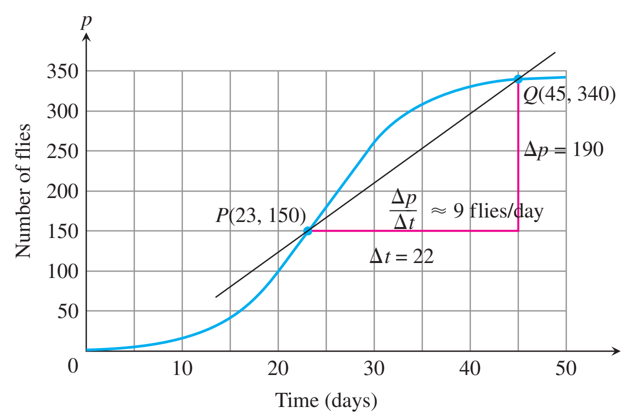


Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

EXAMPLE 2 Growing Drosophila in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued

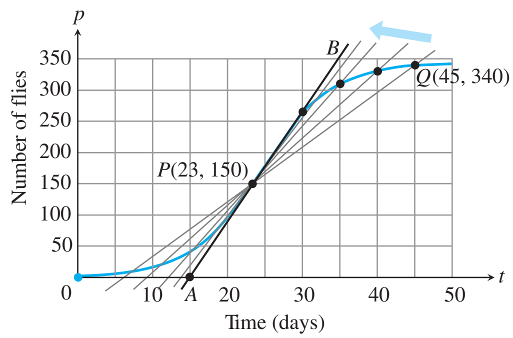
This average rate of change is also the slope of the secant line through the two points P and Q on the population curve. We can calculate the slope of the secant PQ from the coordinates of P and Q.

$$\textit{Secant slope:}\quad \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}$$

Now try Exercise 7.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant PQ change as we back Q along the curve toward P. The results for four positions of Q are shown in Figure 2.28.



(a)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	(340 – 150)/(45 – 23) ≈ 8.6
(40, 330)	(330 – 150)/(40 – 23) ≈ 10.6
(35, 310)	(310 – 150)/(35 – 23) ≈ 13.3
(30, 265)	(265 – 150)/(30 – 23) ≈ 16.4

(b)

Figure 2.28 (a) Four secants to the fruit fly graph of Figure 2.27, through the point $P(23, 150)$. (b) The slopes of the four secants.

In terms of geometry, what we see as Q approaches P along the curve is this: The secant PQ approaches the tangent line AB that we drew by eye at P. This means that within the limitations of our drawing, the slopes of the secants approach the slope of the tangent, which we calculate from the coordinates of A and B to be

$$\frac{350 - 0}{35 - 15} = 17.5 \text{ flies/day.}$$

In terms of population, what we see as Q approaches P is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at P (17.5 flies per day). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day $t = 23$.

Tangent to a Curve

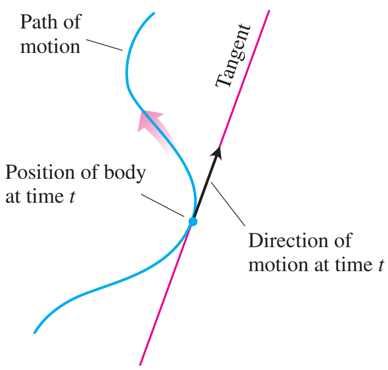
The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the formula $y = f(x)$? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century's major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

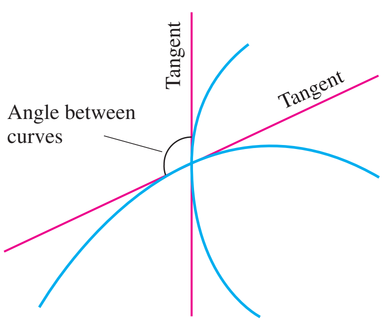
1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

Why Find Tangents to Curves?

In mechanics, the tangent determines the direction of a body's motion at every point along its path.



In geometry, the tangents to two curves at a point of intersection determine the angle at which the curves intersect.



In optics, the tangent determines the angle at which a ray of light enters a curved lens (more about this in Section 3.7). The problem of how to find a tangent to a curve became the dominant mathematical problem of the early seventeenth century and it is hard to overestimate how badly the scientists of the day wanted to know the answer. Descartes went so far as to say that the problem was the most useful and most general problem not only that he knew but that he had any desire to know.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

EXAMPLE 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

SOLUTION

We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$ on the curve (Figure 2.29).

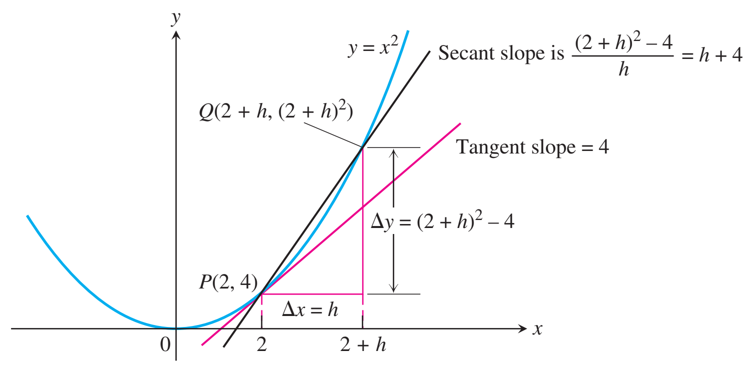


Figure 2.29 The slope of the tangent to the parabola $y = x^2$ at $P(2, 4)$ is 4.

We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P along the curve.

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{h}$$
$$= \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4$$

The limit of the secant slope as Q approaches P along the curve is

$$\lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (h + 4) = 4.$$

Thus, the slope of the parabola at P is 4.

The tangent to the parabola at P is the line through $P(2, 4)$ with slope $m = 4$.

$$y - 4 = 4(x - 2)$$
$$y = 4x - 8 + 4$$
$$y = 4x - 4$$

Now try Exercise 11 (a, b).

Pierre de Fermat

(1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the seventeenth century's major contributions to calculus.

Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$ we use the same dynamic procedure. We calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.30). If the limit exists, it is the slope of the curve at P and we define the tangent at P to be the line through P having this slope.

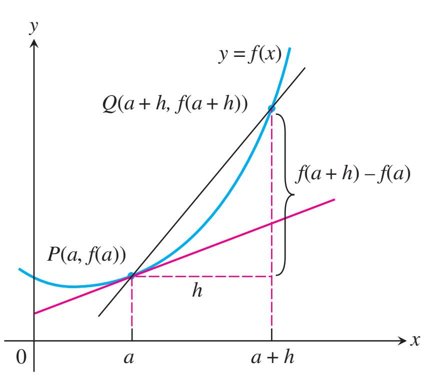


Figure 2.30 The tangent slope is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

DEFINITION Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

EXAMPLE 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.
(b) Where does the slope equal $-1/4$?
(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

SOLUTION

(a) The slope at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \cdot \frac{-h}{ha(a+h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

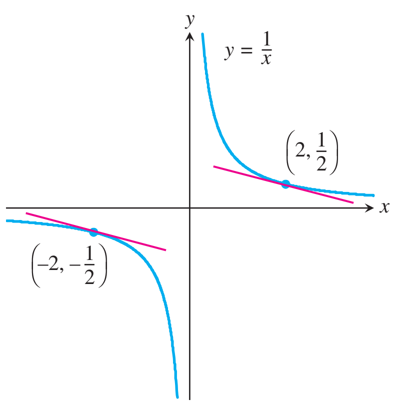


Figure 2.31 The two tangent lines to $y = 1/x$ having slope $-1/4$. (Example 4)

(b) The slope will be $-1/4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4 \qquad \text{Multiply by } -4a^2.$$

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.31).

(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal. ***Now try Exercise 19.***

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of f at a.** Suppose the difference quotient has a limit as h approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function's rate of change with respect to x at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 3.

All of these are the same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

EXAMPLE 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

SOLUTION

The slope of the tangent to the curve at $x = 1$ is

$$\begin{aligned}\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0} \frac{4 - (1+h)^2 - 3}{h} \\ &= \lim_{h\to 0} \frac{4 - 1 - 2h - h^2 - 3}{h} \\ &= \lim_{h\to 0} \frac{-h(2+h)}{h} = -2.\end{aligned}$$

Thus, the slope of the normal is $1/2$, the negative reciprocal of -2. The normal to the curve at $(1, f(1)) = (1, 3)$ is the line through $(1, 3)$ with slope $m = 1/2$.

$$\begin{aligned}y - 3 &= \frac{1}{2}(x - 1) \\ y &= \frac{1}{2}x - \frac{1}{2} + 3 \\ y &= \frac{1}{2}x + \frac{5}{2}\end{aligned}$$

You can support this result by drawing the graphs in a square viewing window.

Now try Exercise 11 (c, d).

About the Word *Normal*

When analytic geometry was developed in the seventeenth century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The Latin word *normalis,* which scholars used for *perpendicular,* became *normal* when they discussed geometry in English.

Speed Revisited

The function $y = 16t^2$ that gave the distance fallen by the rock in Example 1, Section 2.1, was the rock's *position function.* A body's average speed along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its *position* $y = f(t)$. Its **instantaneous speed** at any time t is the **instantaneous rate of change** of position with respect to time at time t, or

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

We saw in Example 1, Section 2.1, that the rock's instantaneous speed at $t = 2$ sec was 64 ft/sec.

Particle Motion

We only have considered objects moving in one direction in this chapter. In Chapter 3, we will deal with more complicated motion.

EXAMPLE 6 Investigating Free Fall

Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.

SOLUTION

The position function of the rock is $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ sec was

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h + 2) = 32 \text{ ft/sec}.$$

Now try Exercise 27.

Quick Review 2.4 *(For help, go to Section 1.1.)*

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2)$, $B(3, 5)$ $\Delta x = 8$, $\Delta y = 3$

2. $A(1, 3)$, $B(a, b)$ $\Delta x = a - 1$, $\Delta y = b - 3$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3)$, $(5, -1)$ Slope $= -\frac{4}{7}$

4. $(-3, -1)$, $(3, 3)$ Slope $= \frac{2}{3}$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$ $y = \frac{3}{2}x + 6$

6. through $(1, 6)$ and $(4, -1)$ $y = -\frac{7}{3}x + \frac{25}{3}$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$ $y = -\frac{3}{4}x + \frac{19}{4}$

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$ $y = \frac{4}{3}x + \frac{8}{3}$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$ $y = \frac{2}{-3}x + \frac{7}{3}$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$? $b = \frac{19}{3}$

Section 2.4 Exercises

In Exercises 1–6, find the average rate of change of the function over each interval.

1. $f(x) = x^3 + 1$ **(a)** 19 **(b)** 1
(a) $[2, 3]$ **(b)** $[-1, 1]$

2. $f(x) = \sqrt{4x + 1}$ **(a)** 1 **(b)** $\frac{7 - \sqrt{41}}{2} \approx 0.298$
(a) $[0, 2]$ **(b)** $[10, 12]$

3. $f(x) = e^x$
(a) $[-2, 0]$ **(b)** $[1, 3]$

4. $f(x) = \ln x$
(a) $[1, 4]$ **(b)** $[100, 103]$

5. $f(x) = \cot t$ **(a)** $-\frac{4}{\pi} \approx -1.273$ **(b)** $-\frac{3\sqrt{3}}{\pi} \approx -1.654$
(a) $[\pi/4, 3\pi/4]$ **(b)** $[\pi/6, \pi/2]$

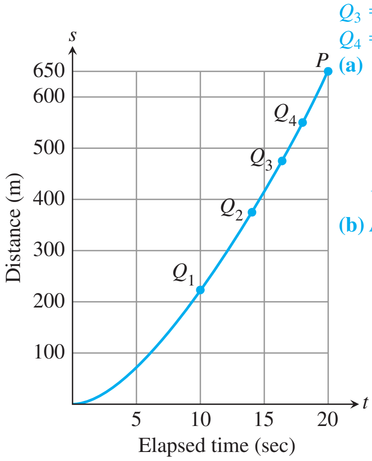
6. $f(x) = 2 + \cos t$ **(a)** $-\frac{2}{\pi} \approx -0.637$ **(b)** 0
(a) $[0, \pi]$ **(b)** $[-\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

(a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What is the appropriate unit for these slopes?

(b) Estimate the speed at point P.

7. ***Accelerating from a Standstill*** The figure shows the distance-time graph for a 1994 Ford® Mustang Cobra™ accelerating from a standstill.



Using $Q_1 = (10, 225)$, $Q_2 = (14, 375)$, $Q_3 = (16.5, 475)$, $Q_4 = (18, 550)$, and $P = (20, 650)$

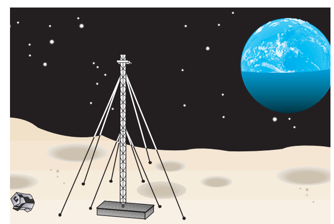
(a)

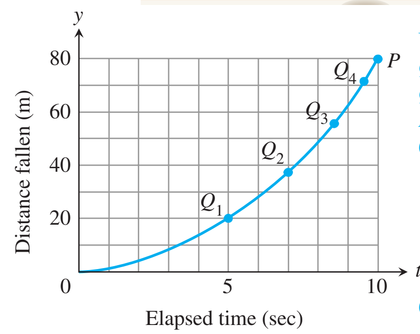
Secant	Slope
PQ_1	43
PQ_2	46
PQ_3	50
PQ_4	50

Units are meters/second

(b) Approximately 50 m/sec

8. ***Lunar Data*** The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.





Using $Q_1 = (5, 20)$, $Q_2 = (7, 38)$, $Q_3 = (8.5, 56)$, $Q_4 = (9.5, 72)$, and $P = (10, 80)$

(a)

Secant	Slope
PQ_1	12
PQ_2	14
PQ_3	16
PQ_4	16

Units are meters/second

(b) Approximately 16 m/sec

In Exercises 9–12, at the indicated point find

(a) the slope of the curve,

(b) an equation of the tangent, and

(c) an equation of the normal.

(d) Then draw a graph of the curve, tangent line, and normal line in the same square viewing window.

9. $y = x^2$ at $x = -2$

10. $y = x^2 - 4x$ at $x = 1$

11. $y = \frac{1}{x - 1}$ at $x = 2$

12. $y = x^2 - 3x - 1$ at $x = 0$

In Exercises 13 and 14, find the slope of the curve at the indicated point.

13. $f(x) = |x|$ at **(a)** $x = 2$ **(b)** $x = -3$ **(a)** 1 **(b)** -1

14. $f(x) = |x - 2|$ at $x = 1$ -1

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.

15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \ge 0 \end{cases}$ at $x = 0$

No. Slope from the left is -2; slope from the right is 2. The two-sided limit of the difference quotient doesn't exist.

16. $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - x, & x \ge 0 \end{cases}$ at $x = 0$ Yes. The slope is -1.

3. (a) $\frac{1 - e^{-2}}{2} \approx 0.432$ **(b)** $\frac{e^3 - e}{2} \approx 8.684$ **4. (a)** $\frac{\ln 4}{3} \approx 0.462$ **(b)** $\frac{\ln(103/100)}{3} = \frac{\ln 1.03}{3} \approx 0.0099$

18. No. The function is discontinuous at $x = \frac{3\pi}{4}$. The left-hand limit of the difference quotient doesn't exist.

17. $f(x) = \begin{cases} 1/x, & x \le 2 \\ \dfrac{4-x}{4}, & x > 2 \end{cases}$ at $x = 2$ Yes. The slope is $-\frac{1}{4}$.

18. $f(x) = \begin{cases} \sin x, & 0 \le x < 3\pi/4 \\ \cos x, & 3\pi/4 \le x \le 2\pi \end{cases}$ at $x = 3\pi/4$

In Exercises 19–22, **(a)** find the slope of the curve at $x = a$.
(b) Writing to Learn Describe what happens to the tangent at $x = a$ as a changes.

19. $y = x^2 + 2$ **(a)** $2a$ **(b)** The slope of the tangent steadily increases as a increases.

20. $y = 2/x$ **(a)** $-\frac{2}{a^2}$ **(b)** The slope of the tangent is always negative. The tangents are very steep near $x = 0$ and nearly horizontal as a moves away from the origin.

21. $y = \dfrac{1}{x-1}$ **(a)** $-\frac{1}{(a-1)^2}$ **(b)** The slope of the tangent is always negative. The tangents are very steep near $x = 1$ and nearly horizontal as a moves away from the origin.

22. $y = 9 - x^2$
(a) $-2a$ **(b)** The slope of the tangent steadily decreases as a increases.

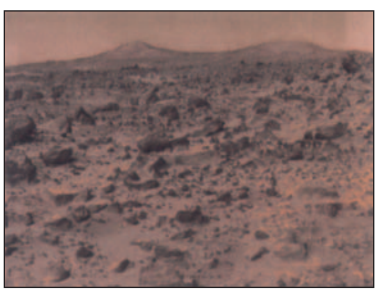
23. *Free Fall* An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped? 19.6 m/sec

24. *Rocket Launch* At t sec after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec? 60 ft/sec

25. *Area of Circle* What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.? 6π in²/in.

26. *Volume of Sphere* What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.? 16π in³/in.

27. *Free Fall on Mars* The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec. 3.72 m/sec



28. *Free Fall on Jupiter* The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec. 45.76 m/sec

29. *Horizontal Tangent* At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal? $(-2, -5)$

30. *Horizontal Tangent* At what point is the tangent to $f(x) = 3 - 4x - x^2$ horizontal? $(-2, 7)$

31. *Finding Tangents and Normals*

(a) Find an equation for each tangent to the curve $y = 1/(x-1)$ that has slope -1. (See Exercise 21.) At $x = 0$: $y = -x - 1$ At $x = 2$: $y = -x + 3$

(b) Find an equation for each normal to the curve $y = 1/(x-1)$ that has slope 1. At $x = 0$: $y = x - 1$ At $x = 2$: $y = x - 1$

32. *Finding Tangents* Find the equations of all lines tangent to $y = 9 - x^2$ that pass through the point (1, 12).
At $x = -1$: $y = 2x + 10$ At $x = 3$: $y = -6x + 18$

33. Table 2.2 gives the amount of federal spending in billions of dollars for national defense for several years.

Table 2.2 National Defense Spending

Year	National Defense Spending ($ billions)
1990	299.3
1995	272.1
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(e) 1990 to 1995: −11.4 billion dollars per year; 2000 to 2001: 23.4 billion dollars per year; 2002 to 2003: 32.1 billion dollars per year

(a) Find the average rate of change in spending from 1990 to 1995. −5.4 billion dollars per year

(b) Find the average rate of change in spending from 2000 to 2001. 11.0 billion dollars per year

(c) Find the average rate of change in spending from 2002 to 2003. 56.3 billion dollars per year

(d) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Find the quadratic regression equation for the data and superimpose its graph on a scatter plot of the data. $y \approx 2.177x^2 - 22.315x + 306.443$

(e) Compute the average rates of change in parts (a), (b), and (c) using the regression equation.

(f) Use the regression equation to find how fast the spending was growing in 2003. 34.3 billion dollars per year

(g) Writing to Learn Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on this equation.
One possible reason is that the war in Iraq and increased spending to prevent terrorist attacks in the U.S. caused an unusual increase in defense spending.

34. Table 2.3 gives the amount of federal spending in billions of dollars for agriculture for several years.

Table 2.3 Agriculture Spending

Year	Agriculture Spending ($ billions)
1990	12.0
1995	9.8
1999	23.0
2000	36.6
2001	26.4
2002	22.0
2003	22.6

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot of the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 2000, Q_2 the point corresponding to 2001, and Q_3 the point corresponding to 2002. Find the slope of the secant line PQ_i for $i = 1, 2, 3$.
Slope of $PQ_1 = -4.7$, Slope of $PQ_2 = -1.9$, Slope of $PQ_3 = 0.6$.

41. (a) $\frac{e^{1+h} - e}{h}$ **(b)** Limit ≈ 2.718 **(c)** They're about the same. **(d)** Yes, it has a tangent whose slope is about e.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

35. True or False If the graph of a function has a tangent line at $x = a$, then the graph also has a normal line at $x = a$. Justify your answer. True. The normal line is perpendicular to the tangent line at the point.

36. True or False The graph of $f(x)=|x|$ has a tangent line at $x = 0$. Justify your answer.
False. There's no tangent at $x = 0$ because f has no slope at $x = 0$.

37. Multiple Choice If the line L tangent to the graph of a function f at the point (2, 5) passes through the point (−1, −3), what is the slope of L? D

(A) −3/8 **(B)** 3/8 **(C)** −8/3 **(D)** 8/3 **(E)** undefined

38. Multiple Choice Find the average rate of change of $f(x) = x^2 + x$ over the interval [1, 3]. E

(A) −5 **(B)** 1/5 **(C)** 1/4 **(D)** 4 **(E)** 5

39. Multiple Choice Which of the following is an equation of the tangent to the graph of $f(x) = 2/x$ at $x = 1$? C

(A) $y = -2x$ **(B)** $y = 2x$ **(C)** $y = -2x + 4$

(D) $y = -x + 3$ **(E)** $y = x + 3$

40. Multiple Choice Which of the following is an equation of the normal to the graph of $f(x) = 2/x$ at $x = 1$? A

(A) $y = \frac{1}{2}x + \frac{3}{2}$ **(B)** $y = -\frac{1}{2}x$ **(C)** $y = \frac{1}{2}x + 2$

(D) $y = -\frac{1}{2}x + 2$ **(E)** $y = 2x + 5$

Explorations

In Exercises 41 and 42, complete the following for the function.

(a) Compute the difference quotient

$$\frac{f(1 + h) - f(1)}{h}.$$

(b) Use graphs and tables to estimate the limit of the difference quotient in part (a) as $h \to 0$.

(c) Compare your estimate in part (b) with the given number.

(d) Writing to Learn Based on your computations, do you think the graph of f has a tangent at $x = 1$? If so, estimate its slope. If not, explain why not.

41. $f(x) = e^x$, e

42. $f(x) = 2^x$, ln 4

Group Activity In Exercises 43–46, the curve $y = f(x)$ has a **vertical tangent** at $x = a$ if

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \infty$$

or if

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = -\infty.$$

In each case, the right- and left-hand limits are required to be the same: both $+\infty$ or both $-\infty$.

Use graphs to investigate whether the curve has a vertical tangent at $x = 0$.

43. $y = x^{2/5}$ No

44. $y = x^{3/5}$ Yes

45. $y = x^{1/3}$ Yes

46. $y = x^{2/3}$ No

Extending the Ideas

In Exercises 47 and 48, determine whether the graph of the function has a tangent at the origin. Explain your answer.

47. $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

48. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

49. *Sine Function* Estimate the slope of the curve $y = \sin x$ at $x = 1$. (*Hint:* See Exercises 41 and 42.) Slope ≈ 0.540

42. (a) $\frac{2^{1+h} - 2}{h}$ **(b)** Limit ≈ 1.386 **(c)** They're about the same. **(d)** Yes, it has a tangent whose slope is about 4 ln.

Quick Quiz for AP* Preparation: Sections 2.3 and 2.4

You may use a calculator with these problems.

1. Multiple Choice Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval (0, 3)? D

(A) −3 **(B)** −1 **(C)** −1/3 **(D)** 1/3 **(E)** 3

2. Multiple Choice Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \le x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \le 6 \\ 1, & 6 < x < 8 \end{cases}$$? E

(A) $\lim_{x\to 4} f(x)$ exists **(B)** $f(4)$ exists

(C) $\lim_{x\to 6} f(x)$ exists **(D)** $\lim_{x\to 8^-} f(x)$ exists

(E) f is continuous at $x = 4$

3. Multiple Choice Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$? B

(A) $y = \frac{1}{4}x + \frac{9}{2}$ **(B)** $y = -4x + 13$

(C) $y = -4x - 3$ **(D)** $y = 4x - 3$

(E) $y = 4x + 13$

4. Free Response Let $f(x) = 2x - x^2$.

(a) Find $f(3)$. −3 **(b)** Find $f(3 + h)$. $-3 - 4h - h^2$

(c) Find $\frac{f(3 + h) - f(3)}{h}$. $-4 - h$

(d) Find the instantaneous rate of change of f at $x = 3$. −4

Chapter 2 Key Terms

average rate of change (p. 87)
average speed (p. 59)
connected graph (p. 83)
Constant Multiple Rule for Limits (p. 61)
continuity at a point (p. 78)
continuous at an endpoint (p. 79)
continuous at an interior point (p. 79)
continuous extension (p. 81)
continuous function (p. 81)
continuous on an interval (p. 81)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
end behavior model (p. 74)
free fall (p. 91)
horizontal asymptote (p. 70)
infinite discontinuity (p. 80)
instantaneous rate of change (p. 91)
instantaneous speed (p. 91)
intermediate value property (p. 83)
Intermediate Value Theorem for Continuous Functions (p. 83)
jump discontinuity (p. 80)
left end behavior model (p. 74)
left-hand limit (p. 64)
limit of a function (p. 60)
normal to a curve (p. 91)
oscillating discontinuity (p. 80)
point of discontinuity (p. 79)
Power Rule for Limits (p. 71)
Product Rule for Limits (p. 61)
Properties of Continuous Functions (p. 82)
Quotient Rule for Limits (p. 61)
removable discontinuity (p. 80)
right end behavior model (p. 74)
right-hand limit (p. 64)
Sandwich Theorem (p. 65)
secant to a curve (p. 87)
slope of a curve (p. 89)
Sum Rule for Limits (p. 61)
tangent line to a curve (p. 88)
two-sided limit (p. 64)
vertical asymptote (p. 72)
vertical tangent (p. 94)

Chapter 2 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, find the limits.

1. $\lim_{x\to -2} (x^3 - 2x^2 + 1)$ -15

2. $\lim_{x\to -2} \dfrac{x^2+1}{3x^2 - 2x + 5}$ $\dfrac{5}{21}$

3. $\lim_{x\to 4} \sqrt{1-2x}$ No limit

4. $\lim_{x\to 5} \sqrt[4]{9 - x^2}$ No limit

5. $\lim_{x\to 0} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$ $-\dfrac{1}{4}$

6. $\lim_{x\to \pm\infty} \dfrac{2x^2+3}{5x^2+7}$ $\dfrac{2}{5}$

7. $\lim_{x\to \pm\infty} \dfrac{x^4 + x^3}{12x^3 + 128}$ $+\infty, \ -\infty$

8. $\lim_{x\to 0} \dfrac{\sin 2x}{4x}$ $\dfrac{1}{2}$

9. $\lim_{x\to 0} \dfrac{x \csc x + 1}{x \csc x}$ 2

10. $\lim_{x\to 0} e^x \sin x$ 0

11. $\lim_{x\to 7/2^+} \text{int}\,(2x-1)$ 6

12. $\lim_{x\to 7/2^-} \text{int}\,(2x-1)$ 5

13. $\lim_{x\to \infty} e^{-x} \cos x$ 0

14. $\lim_{x\to \infty} \dfrac{x + \sin x}{x + \cos x}$ 1

In Exercises 15–20, determine whether the limit exists on the basis of the graph of $y = f(x)$. The domain of f is the set of real numbers.

15. $\lim_{x\to d} f(x)$ Limit exists

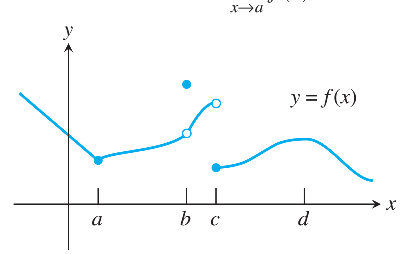
16. $\lim_{x\to c^+} f(x)$ Limit exists

17. $\lim_{x\to c^-} f(x)$ Limit exists

18. $\lim_{x\to c} f(x)$ Doesn't exist

19. $\lim_{x\to b} f(x)$ Limit exists

20. $\lim_{x\to a} f(x)$ Limit exists



In Exercises 21–24, determine whether the function f used in Exercises 15–20 is continuous at the indicated point.

21. $x = a$ Yes

22. $x = b$ No

23. $x = c$ No

24. $x = d$ Yes

In Exercises 25 and 26, use the graph of the function with domain $-1 \le x \le 3$.

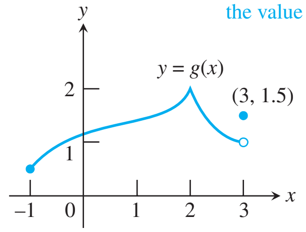
25. Determine

(a) $\lim_{x\to 3^-} g(x)$. 1 **(b)** $g(3)$. 1.5

(c) whether $g(x)$ is continuous at $x = 3$. No

(d) the points of discontinuity of $g(x)$. g is discontinuous at $x = 3$ (and points not in domain).

(e) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not. Yes, can remove discontinuity at $x = 3$ by assigning the value 1 to $g(3)$.



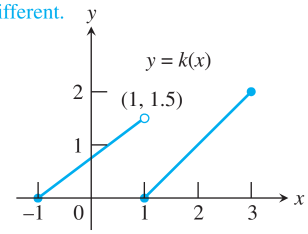
26. Determine

(a) $\lim_{x\to 1^-} k(x)$. 1.5 **(b)** $\lim_{x\to 1^+} k(x)$. 0 **(c)** $k(1)$. 0

(d) whether $k(x)$ is continuous at $x = 1$. No

(e) the points of discontinuity of $k(x)$. k is discontinuous at $x = 1$ (and points not in domain).

(f) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not. Discontinuity at $x = 1$ is not removable because the two one-sided limits are different.



In Exercises 27 and 28, **(a)** find the vertical asymptotes of the graph of $y = f(x)$, and **(b)** describe the behavior of $f(x)$ to the left and right of any vertical asymptote.

27. $f(x) = \dfrac{x+3}{x+2}$ **28.** $f(x) = \dfrac{x-1}{x^2(x+2)}$

In Exercises 29 and 30, answer the questions for the piecewise-defined function.

29. $f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = -1$, 0, and 1.

(b) Does f have a limit as x approaches -1? 0? 1? If so, what is it? If not, why not?

(c) Is f continuous at $x = -1$? 0? 1? Explain.

30. $f(x) = \begin{cases} |x^3 - 4x|, & x < 1 \\ x^2 - 2x - 2, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = 1$. Left-hand limit = 3 Right-hand limit = -3

(b) Does f have a limit as $x \to 1$? If so, what is it? If not, why not? No, because the two one-sided limits are different.

(c) At what points is f continuous? Every place except for $x = 1$

(d) At what points is f discontinuous? At $x = 1$

In Exercises 31 and 32, find all points of discontinuity of the function.

31. $f(x) = \dfrac{x+1}{4-x^2}$ $x = -2$ and $x = 2$ **32.** $g(x) = \sqrt[3]{3x+2}$ There are no points of discontinuity.

In Exercises 33–36, find **(a)** a power function end behavior model and **(b)** any horizontal asymptotes.

33. $f(x) = \dfrac{2x+1}{x^2-2x+1}$ **(a)** $2/x$ **(b)** $y = 0$ (x-axis) **34.** $f(x) = \dfrac{2x^2+5x-1}{x^2+2x}$ **(a)** 2 **(b)** $y = 2$

35. $f(x) = \dfrac{x^3-4x^2+3x+3}{x-3}$ **(a)** x^2 **(b)** None **36.** $f(x) = \dfrac{x^4-3x^2+x-1}{x^3-x+1}$ **(a)** x **(b)** None

In Exercises 37 and 38, find **(a)** a right end behavior model and **(b)** a left end behavior model for the function.

37. $f(x) = x + e^x$ **(a)** e^x **(b)** x **38.** $f(x) = \ln|x| + \sin x$ **(a)** $\ln|x|$ **(b)** $\ln|x|$

Group Activity In Exercises 39 and 40, what value should be assigned to k to make f a continuous function?

39. $f(x) = \begin{cases} \dfrac{x^2+2x-15}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$ $k = 8$

40. $f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \ne 0 \\ k, & x = 0 \end{cases}$ $k = \dfrac{1}{2}$

Group Activity In Exercises 41 and 42, sketch a graph of a function f that satisfies the given conditions.

41. $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = \infty$,

$\lim_{x\to 3^+} f(x) = \infty$, $\lim_{x\to 3^-} f(x) = -\infty$

42. $\lim_{x\to 2} f(x)$ does not exist, $\lim_{x\to 2^+} f(x) = f(2) = 3$

43. *Average Rate of Change* Find the average rate of change of $f(x) = 1 + \sin x$ over the interval $[0, \pi/2]$. $\dfrac{2}{\pi}$

44. *Rate of Change* Find the instantaneous rate of change of the volume $V = (1/3)\pi r^2 H$ of a cone with respect to the radius r at $r = a$ if the height H does not change. $\dfrac{2}{3}\pi aH$

45. *Rate of Change* Find the instantaneous rate of change of the surface area $S = 6x^2$ of a cube with respect to the edge length x at $x = a$. $12a$

46. *Slope of a Curve* Find the slope of the curve $y = x^2 - x - 2$ at $x = a$. $2a - 1$

47. *Tangent and Normal* Let $f(x) = x^2 - 3x$ and $P = (1, f(1))$. Find **(a)** the slope of the curve $y = f(x)$ at P, **(b)** an equation of the tangent at P, and **(c)** an equation of the normal at P.

(a) -1 **(b)** $y = -x - 1$ **(c)** $y = x - 3$

48. ***Horizontal Tangents*** At what points, if any, are the tangents to the graph of $f(x) = x^2 - 3x$ horizontal? (See Exercise 47.) $\left(\frac{3}{2}, -\frac{9}{4}\right)$

49. ***Bear Population*** The number of bears in a federal wildlife reserve is given by the population equation

$$p(t) = \frac{200}{1 + 7e^{-0.1t}},$$

where t is in years.

(a) **Writing to Learn** Find $p(0)$. Give a possible interpretation of this number. 25. Perhaps this is the number of bears placed in the reserve when it was established.

(b) Find $\lim_{t\to\infty} p(t)$. 200

(c) **Writing to Learn** Give a possible interpretation of the result in part (b).

50. ***Taxi Fares*** Bluetop Cab charges \$3.20 for the first mile and \$1.35 for each additional mile or part of a mile.

(a) Write a formula that gives the charge for x miles with $0 \le x \le 20$. $f(x) = \begin{cases} 3.20 - 1.35 \times \text{int}(-x + 1), & 0 < x \le 20 \\ 0, & x = 0 \end{cases}$

(b) Graph the function in (a). At what values of x is it discontinuous? f is discontinuous at integer values of x: 0, 1, 2, . . . , 19

51. Table 2.4 gives the population of Florida for several years.

Table 2.4 Population of Florida

Year	Population (in thousands)
1998	15,487
1999	15,759
2000	15,983
2001	16,355
2002	16,692
2003	17,019

Source: U.S. Census Bureau, *Statistical Abstract of the United States; 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot for the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 1998, Q_2 the point corresponding to 1999, . . . , and Q_5 the point corresponding to 2002. Find the slope of the secant the PQ_i for $i = 1, 2, 3, 4, 5$.

(c) Predict the rate of change of population in 2003.

(d) Find a linear regression equation for the data, and use it to calculate the rate of the population in 2003.

49. (c) Perhaps this is the maximum number of bears which the reserve can support due to limitations of food, space, or other resources. Or, perhaps the number is capped at 200 and excess bears are moved to other locations.

51. (b) Slope of $PQ_1 = 306.4$; slope of $PQ_2 = 315$; slope of $PQ_3 = 345.3$; slope of $PQ_4 = 332$; slope of $PQ_5 = 327$

(c) We use the average rate of change in the population from 2002 to 2003 which is 327,000.

(d) $y \approx 309.457x + 12966.533$, rate of change is 309 thousand because rate of change of a linear function is its slope.

52. ***Limit Properties*** Assume that

$$\lim_{x\to c} [f(x) + g(x)] = 2,$$

$$\lim_{x\to c} [f(x) - g(x)] = 1,$$

and that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Find $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$.
$\lim_{x\to c} f(x) = 3/2$; $\lim_{x\to c} g(x) = 1/2$

AP* Examination Preparation

You should solve the following problems without using a graphing calculation.

53. **Free Response** Let $f(x) = \dfrac{x}{|x^2 - 9|}$. All real numbers except 3 or −3.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote of the graph of f. $x = -3$ and $x = 3$

(c) Write an equation for each horizontal asymptote of the graph of f. $y = 0$

(d) Is f odd, even, or neither? Justify your answer.

(e) Find all values of x for which f is discontinuous and classify each discontinuity as removable or nonremovable.
$x = -3$ and $x = 3$. Both are nonremovable.

54. **Free Response** Let $f(x) = \begin{cases} x^2 - a^2x & \text{if } x < 2, \\ 4 - 2x^2 & \text{if } x \ge 2. \end{cases}$

(a) Find $\lim_{x\to 2^-} f(x)$. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - a^2x) = 4 - 2a^2$.

(b) Find $\lim_{x\to 2^+} f(x)$. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4 - 2x^2) = -4$

(c) Find all values of a that make f continuous at 2. Justify your answer.

55. **Free Response** Let $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3}$.

(a) Find all zeros of f.

(b) Find a right end behavior model $g(x)$ for f. $g(x) = x$.

(c) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.
$\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{x^3 - 2x^2 + 1}{x^3 + 3x} = 1.$

53. (d) Odd, because $f(-x) = \dfrac{-x}{|(-x)^2 - 9|} = \dfrac{x}{|x^2 - 9|} = -f(x)$ for all x in the domain.

54. (c) For $\lim_{x\to 2} f(x)$ to exist, we must have $4 - 2a^2 = -4$, so $a = \pm 2$. If $a = \pm 2$, then $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = -4$, making f continuous at 2 by definition.

55. (a) The zeros of $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3}$ are the same as the zeros of the polynomial $x^3 - 2x^2 + 1$. By inspection, one such zero is $x = 1$. Divide $x^3 - 2x^2 + 1$ by $x - 1$ to get $x^2 - x^2 - 1$, which has zeros $\dfrac{1 \pm \sqrt{5}}{2}$ by the quadratic formula. Thus, the zeros of f are 1, $\dfrac{1 + \sqrt{5}}{2}$, and $\dfrac{1 - \sqrt{5}}{2}$.