Geometric Constructions

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"When will we ever use this?" This is a question that every teacher has heard at some point or at several points in time. But a better question would be, "Where has this been used this in the past?" It is important to not only look to the future, but to also look to the past. To fully understand a topic, whether it deals with science, social studies, or mathematics, its history should be explored. Specifically, to fully understand geometric constructions the history is definitely important to learn. As the world progresses and evolves so too does geometry. In high school classrooms today the role of geometry constructions has dramatically changed.

In order to understand the role of geometry today, the history of geometry must be discussed. As Marshall and Rich state in the article, *The Role of History in a Mathematics Class* [8],

> "…history has a vital role to play in today's mathematics classrooms. It allows students and teachers to think and talk about mathematics in meaningful ways. It demythologizes mathematics by showing that it is the creation of human beings. History enriches the mathematics curriculum. It deepens the values and broadens the knowledge that students construct in mathematics class."

This quote truly sums up the importance of relating the past to the present. Students will benefit from knowing about how mathematical topics arose and why they are still important today.

To thoroughly examine the history of geometry, we must go back to ancient Egyptian mathematics. A topic that often amazes people is the beautiful geometry in Egyptian pyramids. The mathematics and specifically geometry involved in the building of these pyramids is extensive. From Egypt, Thales brought geometric ideas and introduced them to Greece. This then led the important evolution of Greek deductive proofs. Thales is known to have come up with five theorems in geometry [14].

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. The vertical angles between two intersecting straight lines are equal.
4. Two triangles are congruent if they have two angles and one side equal.
5. An angle in a semicircle is a right angle.

However, the title of the "father of geometry" is often given to Euclid. Living around the time of 300 BC, he is most known for his book *The Elements.* He took the ideas of Thales and other mathematicians and put them down in an organized collection of definitions, axioms and postulates. From these basics, the rest of geometry evolves. In *The Elements*, the first four definitions are as follows:

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.

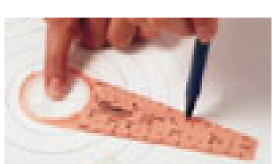
Sir Thomas Heath wrote a respected translation of Euclid's *The Elements* in 1926 entitled *The Thirteen Books of Euclid's Elements* [11]. This translation seems to be the most accepted version of Euclid's writings given modifications and additions.

Since the time of Euclid there have been three famous problems which have captivated the minds and of many mathematicians. These three problems of antiquity are as follows:

1. Squaring the Circle
2. Doubling the Cube
3. Trisecting an Angle.

Far back in history and to this present day, these problems are discussed in detail.

In early geometry, the tools of the trade were a compass and straightedge. A compass was strictly used to make circles of a given radius. Greeks used collapsible compasses, which would automatically collapse. Nowadays, we use rigid compasses, which can hold a certain radius, but is has been shown that construction with rigid compass and straightedge is equivalent to construction with collapsible compass and straightedge. However, compasses have changed dramatically over the years. Some compasses have markings used to construct circles with a given radius. Of course, under the strict rules of Greeks, these compasses would not have been allowed.



More strictly, there were no markings on the straightedge. A straightedge was to be used only for drawing a segment between two points. There were very specific rules about what could and could not be used for mathematical drawings. These drawings, known as constructions, had to be exact. If the rules were broken, the mathematics involved in the constructions was often disregarded. When describing these concepts to students nowadays, showing pictures of ancient paintings with these tools help illustrate the importance and commonplace of geometry and these aforementioned tools.

A portion of Raphaello Sanzio's painting *The School of Athens* from 16th century

The Measurers: A Flemish Image of Mathematics in the 16th century

In regards to the history of constructions, a Danish geometer, Georg Mohr, proved that any construction that could be created by using a compass and straightedge could in fact be created by a compass alone. This surprising fact published in 1672 is normally credited to the Italian mathematician, Lorenzo Mascherone from the eighteenth century. Hence, constructions created using only compasses are called Mascheroni constructions [19].

After Euclid, geometry continued to evolve led by Archimedes, Apollonius and others. However, the next mathematicians to make a dramatic shift in the nature of geometry were the French mathematicians, René Descartes and Pierre de Fermat, in the seventeenth century, who introduced coordinate geometry. This advance of connecting algebra to geometry directly led to other great advances in many areas of mathematics.

Non-Euclidean geometry was the next major movement. János Bolyai, following the footsteps of his father, attempted to create a new axiom to replace Euclid's fifth axiom. Around 1824, this study led to development of a new geometry called non-Euclidean geometry. Another mathematician that made contributions to the formation of non-Euclidean geometry was Nikolai Ivanovich Lobachevsky. In 1840, Lobachevsky published *Géométrie imaginaire* [12]. Because of Bolyai and Lobachevsky's direct connection to Gauss, some believe that non-Euclidean geometry should in fact be credited to Gauss [15].

Even now, geometry continues to progress. In addition, how schools teach geometry has continued to change. In the past, compass and straightedge constructions were a part of the curriculum. However, in most recent years, constructions have faded out. In older textbooks, constructions were entire chapters. However, in newer textbooks, constructions are in the middle of chapters and discussed very briefly.

Instead of concentrating on paper and pencil, compass and straightedge constructions, current books tend to emphasize the use of dynamic computer software, such as Geometer's Sketchpad.   The sloppiness and inaccuracy of man-made constructions could be avoided by the use of technology. Though there are still educators that believe that using this technology is not true geometry, most realize the benefits that such software can have on comprehension.

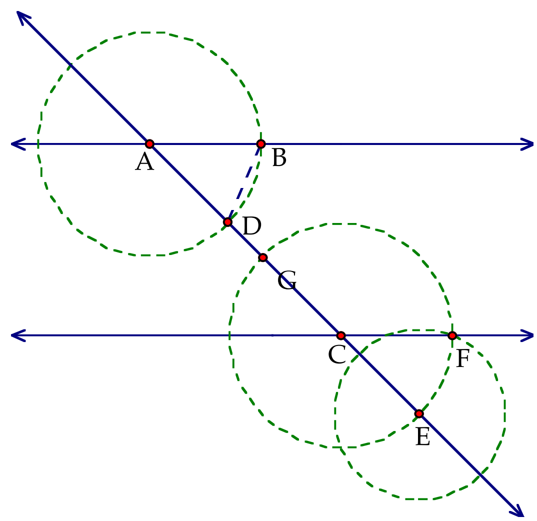
Will true Euclidean constructions using a compass and straightedge on paper soon be a thing of the past? Will it be another lost mathematical concept like finding square roots and logarithms? Will it always been seen as an important link to the past? Will it be recognized as important but is replaced by constructions using technology?

As a link to the past, students might find constructions interesting when related to the three famous problems of antiquity of circle squaring, cube duplicating, and angle trisecting. These problems went unsolved for many years under the Greek rules of constructions. It was not until several hundred years later that they were shown to be impossible using only a compass and straightedge. The mathematics in of circle squaring, cube duplicating, and angle trisecting is interesting and can lead to good discussions.

The basics of constructions must be discussed before the complexity of the three ancient problems can be explained.   The ancient Greeks' way of representing numbers was cumbersome, with no symbol for 0 and no place-value. Perhaps as a consequence, they did arithmetic geometrically. We will use modern notation to analyze what numbers could be constructed by straightedge and compass and to study the three ancient problems.

In order to make arithmetic constructions, two segments, one of length x and the other length y, and a unit length of 1 are given. Through basic geometry and algebra, other related lengths can be created. Five arithmetic constructions are $x+y$, $x-y$, xy, x/y, and $\sqrt{x}$. In order to carry out these arithmetic constructions, we must first be able to construct a parallel line.

Parallel lines:

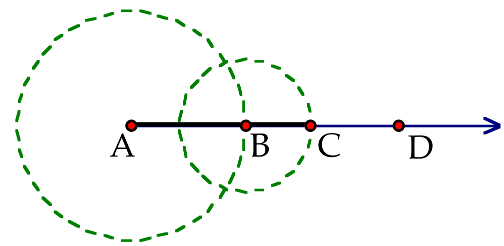


Given: $\overleftrightarrow{AB}$ and point C not on line $\overleftrightarrow{AB}$

Construct: a line parallel to $\overleftrightarrow{AB}$.

1. With a straightedge, draw $\overleftrightarrow{AC}$.
2. With a compass, construct a circle with center at A and a radius of $|AB|$. Let D be the point of intersection of this circle with $\overrightarrow{AC}$.
3. With a compass, construct a circle with center at C and a radius of $|AB|$. Let E be the point of intersection, not between A and C, of this circle with $\overrightarrow{AC}$.
4. With a compass, construct a circle with center at E and a radius of $|DB|$. Let F be the point of intersection of this circle and circle C.
5. With a straightedge, connect points C and F. Then $\overleftrightarrow{AB} \| \overleftrightarrow{CF}$.

Addition:



Given: two lengths x and y

Construct: $x + y$

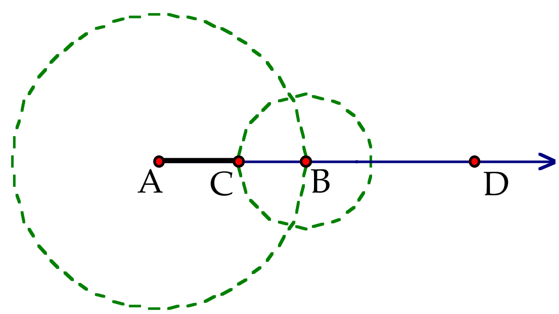
1. With a straightedge, draw $\overrightarrow{AD}$, so that $AD > x + y$.
2. With a compass, construct a circle with center at A and a radius of length x. Let B be the point of intersection of this circle with $\overline{AD}$.
3. With a compass, construct a circle with center at B and a radius of length y. Let C be the point of intersection of this circle with $\overrightarrow{BD}$.
4. With a straightedge, connect points A and C. Then $AC = x + y$.

Thus, the construction of a addition is possible.

Subtraction:

Given: two lengths x and y, where $x > y$

Construct: $x - y$

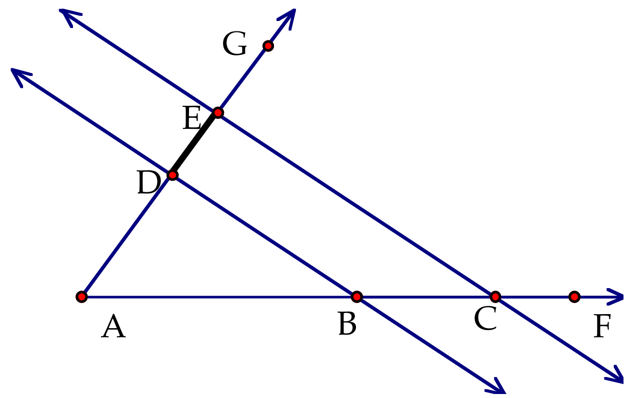


1. With a straightedge, draw $\overrightarrow{AD}$.
2. With a compass, construct a circle with center at A and a radius of length x. Let B be the point of intersection of this circle with $\overrightarrow{AD}$.
3. With a compass, construct a circle with center at B and a radius of length y. Let C be the point of intersection of this circle with $\overline{AB}$.
4. With a straightedge, connect points A and C. Then $AC = x - y$.

Thus, the construction of a difference is possible.

The details of three of the arithmetic constructions, xy, x/y, and $\sqrt{x}$, make use of similar triangles as shown below.

Multiplication:



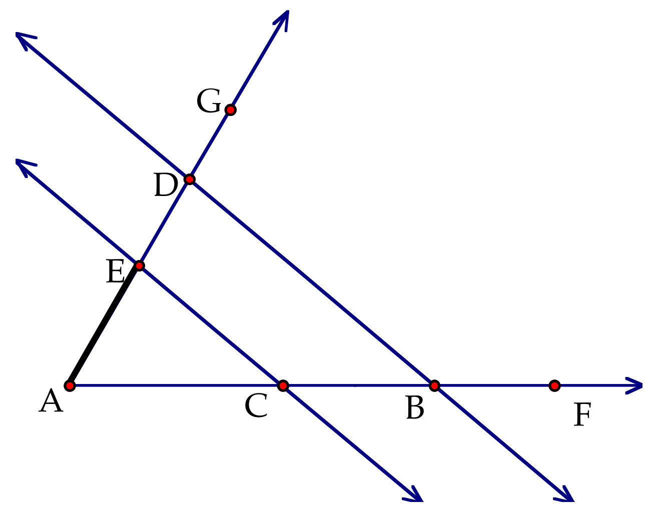
Given: three lengths x, y and unit 1

Construct: xy

1. With a straightedge, draw $\overrightarrow{AF}$ so that $AF > 1 + x$.
2. With a compass, construct a circle with center at A and a radius of length 1. Let B be the point of intersection of this circle with $\overrightarrow{AF}$. $AB = 1$
3. With a compass, construct a circle with center at B and a radius of length x. Let C be the point of intersection of this circle with $\overrightarrow{BF}$. $BC = x$
4. With a straightedge, draw $\overrightarrow{AG}$, with G not on $\overleftrightarrow{AF}$.
5. With a compass, construct a circle with center at A and a radius of length y. Let D be the point of intersection of this circle with $\overrightarrow{AG}$. $AD = y$
6. With a straightedge, construct $\overleftrightarrow{DB}$.
7. Construct the line parallel to $\overleftrightarrow{DB}$ passing through point C.
8. Let E be the intersection of the parallel line and $\overrightarrow{AG}$.
9. With a straightedge, connect points D and E. Then $DE = xy$.

Since $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$, $\Delta ABD \sim \Delta ACE$ by Angle-Angle Similarity. Therefore, the following proportion holds true: $\frac{1}{x} = \frac{y}{DE}$, $DE = xy$. Thus, the construction of a product is possible.

Division:

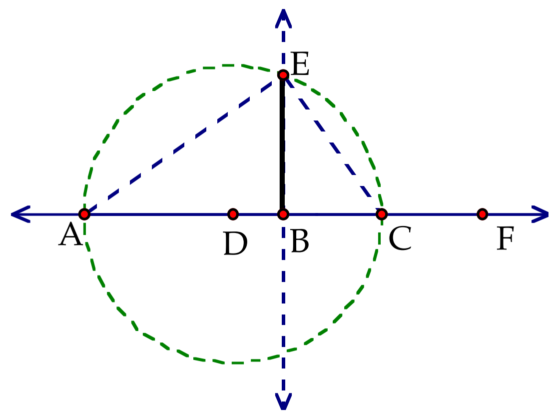


Given: three lengths x, y and unit 1

Construct: y/x

1. With a straightedge, draw $\overrightarrow{AF}$.
2. With a compass, construct a circle with center at A and a radius of length x. Let B be the point of intersection of this circle with $\overrightarrow{AF}$. $AB = x$
3. With a compass, construct a circle with center at A and a radius of length 1. Let C be the point of intersection of this circle with $\overrightarrow{AF}$. $AC = 1$
4. With a straightedge, draw $\overrightarrow{AG}$, with G not on $\overleftrightarrow{AF}$.
5. With a compass, construct a circle with center at A and a radius of length y. Let D be the point of intersection of this circle with $\overrightarrow{AG}$. $AD = y$
6. With a straightedge, construct $\overleftrightarrow{DB}$.
7. Construct the line parallel to $\overleftrightarrow{DB}$ passing through point C.
8. Let E be the intersection of the parallel line and $\overrightarrow{AG}$.
9. With a straightedge, connect points A and E. Then $AE = y/x$.

Since $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$, $\Delta ABD \sim \Delta ACE$ by Angle-Angle Similarity. Therefore, the following proportion holds true: $\dfrac{1}{x} = \dfrac{AE}{y}$, $AE = y/x$. Thus, the construction of a quotient is possible.



Square Root:

Given: two lengths x and unit 1

Construct: $\sqrt{x}$

1. With a straightedge, draw $\overleftrightarrow{AF}$.
2. With a compass, construct a circle with center at A and a radius of length 1. Let B be the point of intersection of this circle with $\overrightarrow{AF}$. $AB = 1$
3. With a compass, construct a circle with center at B and a radius of length x. Let C be the point of intersection of this circle with $\overrightarrow{BF}$. $BC = x$
4. Construct the midpoint D of $\overline{AC}$.
5. With a compass, construct a circle with center at D and a radius of length $|DA|$.
6. Construct a line perpendicular to $\overline{AC}$ passing through point B.
7. Let E be the point of intersection of this perpendicular line and circle D.
8. With a straightedge, connect points B and E. Then $BE = \sqrt{x}$.

Since $\overleftrightarrow{AC} \perp \overleftrightarrow{BE}$ and $\overleftrightarrow{AE} \perp \overleftrightarrow{EC}$, $\Delta ABE \sim \Delta EBC$ by Angle-Angle Similarity. Therefore, the following proportion holds true: $\frac{1}{BE} = \frac{BE}{x}$, $x = BE^2$, $BE = \sqrt{x}$. Thus, the construction of a square root is possible.

These five constructions are crucial to the explanation of why the three geometric problems of antiquity are indeed impossible. Since the rules of addition, subtraction, multiplication, division, and square rooting are possible, the art of constructing numbers using such rules is possible. Numbers constructed using straightedge and compass are called constructible numbers. In terms of field theory, these numbers must lie in certain quadratic extensions of the rationals.

Since only a compass and straightedge can be used, the only constructions that can be created are segments and circles. Since an intersection point is often what is drawn, only an arc of a circle is used and not the entire circle. The construction of new points comes from the intersection of two lines, two circles, or a line and a circle. To find the coordinates of these intersections, the resulting equations would either be linear or quadratic. In either case, the equations are generally simple to solve either using basic arithmetic to solve linear equations or the quadratic formula to solve quadratic equations. Thus, the solution will be a number obtained from given numbers using the basic operations of addition, subtraction, multiplication, division, or taking the square root. All three of the impossible problems of antiquity are unsolvable under Greek construction rules because solutions would not have these characteristics. However, the proofs of showing the impossibility of these problems did not truly come about until the 19^{th} century when geometric concepts could be related to algebraic concepts.

The saying "squaring a circle" has been used throughout the years. The metaphor is used to describe someone trying to attempt something that is impossible. From the most ancient documents, dating back as far as 1550 BC, to more recent documents, the problem of squaring the circle has been recorded. Of the three ancient problems, the most talked about in recent years is the squaring of a circle, sometimes referred to as the quadrature of the circle. This construction entails constructing a square whose area equals that of a given circle. It was not until 1882 that Carl Louis Ferdinand von Lindemann finally proved this to be impossible [13].

To describe this problem in mathematical detail, assume to be given some circle with the radius measuring 1. Therefore the area of the circle is $\pi r^2 = \pi(1)^2 = \pi$. A square with the same area would result in $s^2 = \pi$ therefore $s = \sqrt{\pi}$. In order to construct a square with the same area, the length of a side of the square must be $\sqrt{\pi}$. With the constructions that we know are possible, taking the square root of a number is no problem. However creating a segment with a length of π is a problem since π cannot be created by the simple operations of addition, subtraction, multiplication, or division. It is not debated that a construction can be made ever so close to π. However, a true segment of length π cannot be constructed.

Lindemann proved that π was a transcendental number therefore proving the construction of the number π was impossible. Saying that π is transcendental is the same as saying that π is not the root of any algebraic equation with rational coefficients. Even after Lindemann proved that this construction was impossible, many people still attempted to come up with a way to create π. Many so-called proofs were presented but in the end all of them have been discredited. So in fact it is impossible to construct a square with an area equal to that of a given circle.

The next two problems of antiquity, doubling the cube and trisecting an angle, again are impossible using only a compass and an unmarked straightedge. However, many mathematicians have shown that both constructions are possible if a marked straightedge is used. But under the Greeks' most rigorous rules, only the unmarked straightedge could be used for drawing segments. For both problems, we show that a certain cubic equation does not have rational roots. It then follows that the roots cannot lie in a quadratic extension of the rationals, and so the problem cannot be solved with straightedge and compass.

In keeping with the rules of the Greeks, doubling the cube is constructing a cube with twice the volume as a given cube, of course using only a compass and straightedge. This problem is known as doubling the cube, duplicating the cube, and the Delian problem. During the time of the Greeks this problem was the most famous. However over the years the problem of squaring the circle has overshadowed this now runner-up.

This problem has an interesting history all to itself. Of course the accuracy of these stories themselves has been questioned. The first story is that of Glaucus' tomb that was originally a cube measuring one hundred feet in each direction. Minos was not happy with the size of the tomb and ordered it to be made double the size.

The next and more common story is that of the Delians, which is why this problem is sometimes referred to as the Delian problem. Some say that the problem of doubling the cube originated with this story. Around 430 BC there was a major plague in Athens that in the end claimed the lives of nearly one quarter of the population. During the height of the plague Athenians asked for guidance from the Oracle at Delos as to how to appease the gods so that the plague would come to an end. The Delians were guided to double the size of the altar to the god Apollo. At first the craftsmen thought to double the length of each side of the altar. However, they soon realized that this did not double the size of the altar but in fact it would create an altar eight times the size of the original. After exhausting their ideas, the Delians asked Plato for advice. He responded that the Oracle in fact wanted to embarrass the Greeks for their ignorance of mathematics, primarily their ignorance of geometry. After that time, this problem became so popular that it was studied in detail at Plato's Academy.

Mathematicians attempted to solve the problem with no success. Finally, Hippocrates of Chios showed that the problem was simply the same as finding a solution to $x^3 = 2a^3$, where a is a given segment. Furthermore line segments of x and y may be found such that:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a} \quad \text{which leads to…}$$

$$\begin{aligned} \frac{a^3}{x^3} &= \left(\frac{a}{x}\right)^3 \\ &= \left(\frac{a}{x}\right)\left(\frac{x}{y}\right)\left(\frac{y}{2a}\right) \\ &= \frac{1}{2} \end{aligned}$$

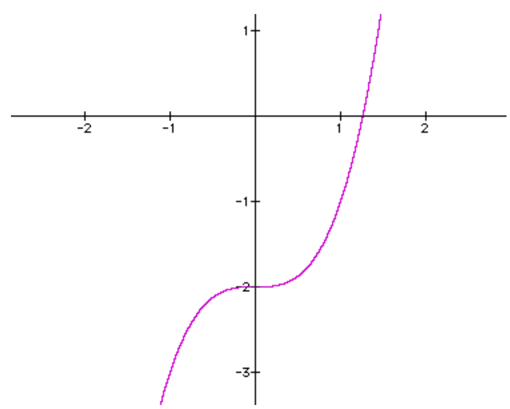
$$\begin{aligned} \frac{a^3}{x^3} &= \frac{1}{2} \\ x^3 &= 2a^3 \end{aligned}$$

When showing the impossibility of doubling the cube using only a compass and straightedge, this information plays an important role.

The impossibility of doubling the cube is equivalent to the impossibility of solving $x^3 - 2 = 0$ with only a compass and straightedge. Linking again back to the history of the Delians, the number $\sqrt[3]{2}$ is sometimes referred to as the Delian constant [20].

A cube of side length one would have a volume of $1^3 = 1$. Doubling the volume would produce a new side length of $\sqrt[3]{2}$ so that the volume would be $\left(\sqrt[3]{2}\right)^3 = 2$. In order to construct a cube with twice the volume, $x^3 - 2 = 0$ must have rational roots. We will show that $x^3 = 2$ is irreducible over the rationals, and thus its roots will not be in any quadratic extension of the rationals.

Assume that $x^3 - 2 = 0$ does have a rational root, $\frac{p}{q}$ where $\frac{p}{q}$ is irreducible. Then



$$\left(\frac{p}{q}\right)^3 - 2 = 0$$
$$\frac{p^3}{q^3} - 2 = 0$$
$$q^3\left(\frac{p^3}{q^3} - 2\right) = q^3(0)$$
$$p^3 - 2q^3 = 0$$
$$p^3 = 2q^3$$

Since $p^3 = 2q^3$ and 2 is prime, 2 divides p. If 2 divides p, then let $p = 2r$.

$$p^3 = 2q^3$$
$$(2r)^3 = 2q^3$$
$$8r^3 = 2q^3$$
$$4r^3 = q^3$$

This implies that 2 divides q also. But this is a contradiction because if that was the case then $\frac{p}{q}$ would have been reducible. Therefore, $x^3 - 2 = 0$ does not have a rational root and hence the solutions are not constructible.

An alternate way to show this is impossible is to use the Rational Roots Theorem [7].

Rational Zero Theorem (Rational Roots Theorem):

If a polynomial function,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0,$$

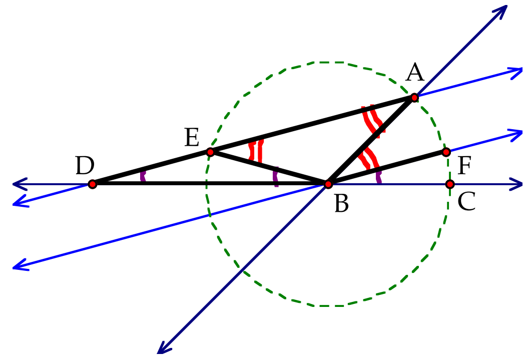
has integer coefficients, then every rational zeros of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

Therefore, the possible rational roots for $x^3 - 2 = 0$ would be ± 1 or ± 2. But none of these are solutions to $x^3 - 2 = 0$. Since, $x^3 - 2 = 0$ has no rational roots, then the solutions to $x^3 - 2 = 0$ are not constructible.

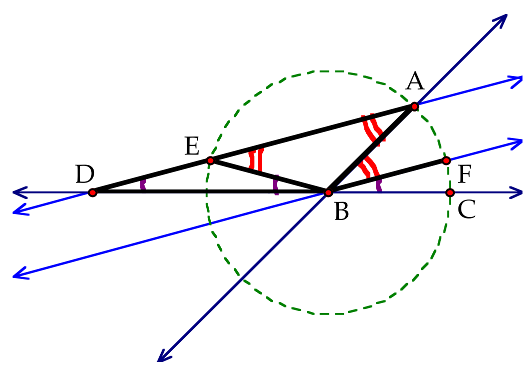
The third historic but probably least popular problem is trisecting an angle. Again, the name describes the problem, dividing a given angle into three smaller angles all of the same measure. One of the most famous trisection of an angle solutions, using a compass and marked straightedge, comes from Archimedes. There are two different constructions that can be completed. However, both use the same overall concepts.

Given: $\angle ABC$ to be trisected



1. Using a compass, construct a circle with center B and radius $|AB|$, where $AB = BC$.
2. **Mark** the distance between A and B.
3. Line up the **marked** straightedge with point A. Let D be the point of intersection of this straightedge and $\overleftrightarrow{BC}$. Let E be the point of intersection of circle A and $\overline{AD}$.
4. Adjust the straightedge until $AB = DE$. *(This is the tricky part.)*
5. Construct a line parallel to $\overleftrightarrow{AD}$ passing through point B. Let F be the point of intersection of this parallel line and circle B.
6. Then $m\angle FBC = \dfrac{m\angle ABC}{3}$

Proof:



Given: $AB = DE$, $m\angle ADB = \alpha$

Prove: $\alpha = \dfrac{m\angle ABC}{3}$

Since $EB = ED$, $m\angle EDB = m\angle EBD$.

Let $m\angle ADB = \alpha$, then $m\angle ADB = m\angle EBD = \alpha$.

In a triangle, the exterior angle is equal to the sum of the two remote interior angles, therefore $m\angle AEB = 2\alpha$.

Since all radii are congruent, $BA = BE$.

Since $BA = BE$, $m\angle AEB = m\angle BAE = 2\alpha$.

Since $\overleftrightarrow{AD} \| \overleftrightarrow{BF}$, $m\angle DAB = m\angle ABF = 2\alpha$. (alternate interior angles are congruent)

and $m\angle ADB = m\angle FBC = \alpha$. (corresponding angles are congruent).

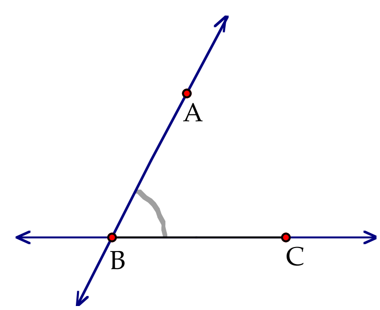
Using the angle addition postulate,

$$\begin{aligned} m\angle ABC &= m\angle ABF + m\angle FBC \\ m\angle ABC &= 2\alpha + \alpha \\ m\angle ABC &= 3\alpha \\ \alpha &= \frac{m\angle ABC}{3} \end{aligned}$$

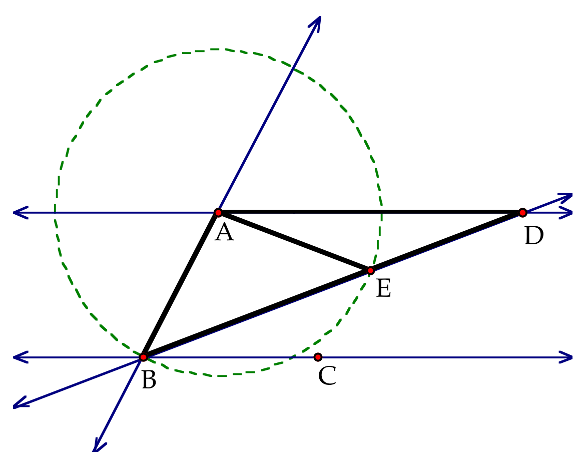
Thus $\angle ABC$ is trisected using a compass and **marked** straightedge.

The next construction uses similar geometric rules but is constructed in a slightly different manner.

Given: $\angle ABC$ to be trisected



1. Construct a line parallel to $\overleftrightarrow{BC}$ passing through point A.
2. Using a compass, construct a circle with center A and radius $|AB|$.
3. **Mark** the distance between A and B.
4. Line up the **marked** straightedge with point B. Let D be the point of intersection of this straightedge and the line parallel to $\overleftrightarrow{BC}$. Let E be the point of intersection of circle A and $\overline{BD}$.

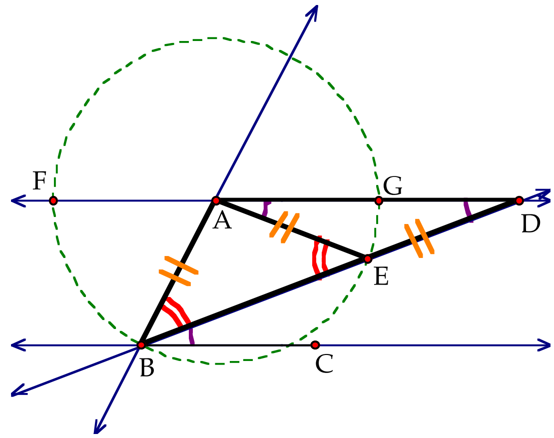


5. Adjust the straightedge until $AB = DE$. *(This is the tricky part.)*
6. Then $m\angle DBC = \dfrac{m\angle ABC}{3}$

Proof:

Given: $AB = DE$

Prove: $m\angle DBC = \dfrac{m\angle ABC}{3}$



Since all radii are congruent, $AB = AE$.

$$m\angle ADB = \frac{1}{2}\left(m\widehat{FB} - m\widehat{GE}\right)$$
$$= \frac{1}{2}\left(m\angle FAB - m\angle GAE\right)$$

Since $DE = AE$, $m\angle GAE = m\angle ADE$.

Furthermore, $\angle ADE$ could also be named $\angle ADB$

By substitution, $m\angle ADB = \frac{1}{2}\left(m\angle FAB - m\angle ADB\right)$.

$$\begin{aligned} m\angle ADB &= \frac{1}{2}\left(m\angle FAB - m\angle ADB\right) \\ 2m\angle ADB &= m\angle FAB - m\angle ADB \\ 3m\angle ADB &= m\angle FAB \\ m\angle ADB &= \frac{m\angle FAB}{3} \end{aligned}$$

Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, alternate interior angles are congruent. $m\angle FAB = m\angle ABC$

Finally, by substitution, $m\angle ADB = \dfrac{m\angle ABC}{3}$.

Again since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, alternate interior angles are congruent. $m\angle ADB = m\angle DBC$

A different method, created by Hippias of Ilis, shows how to trisect an angle using a curve called the Quadratrix of Hippias. Still another method was credited to Hippocrates of Chios. Both methods are accurate but again go against the rigid rules set out by the Greeks, which restricts the tools to only a compass and unmarked straightedge.

A special characteristic of this problem that the others do not have is that some angles can be trisected, while others cannot. In the other two problems it is always impossible to square a circle or double a cube. However, if even one angle cannot be trisected, then we say that the trisection of an angle in general is impossible. Early on, Gauss always claimed that doubling a cube and trisecting an angle were impossible, but gave no proof. However, Pierre Laurent Wantzel proved these problems impossible in an 1837 publication [19]. He showed that the problem of trisecting an angle was the same as solving a cubic equation. Knowing this, he showed that few cubics could be solved using a compass and straightedge only. Therefore, he proved that most angles could not be trisected.

The most common example of an angle that cannot be trisected is 60˚. A 60˚ angle is easily constructed by creating an equilateral triangle. However, trisecting a 60˚ angle is another problem all together. Not only is it not easy, it is not possible. In order to trisect an angle of 60˚, an angle of 20˚ must be able to be constructed. This is the same as constructing the length of the cosine of 20˚. In a unit circle, the lengths of the sides of a right triangle can be given in terms of trigonometric functions of its angles. The horizontal length is cosine of θ and the vertical length is sine of θ. Thus cosine of 20˚ could be constructed if an angle of 20˚ could be constructed, that is, if we could trisect a 60˚ angle. The impossibility of constructing cosine of 20˚ comes from the inability to find a constructible solution to a cubic equation.

The trigonometric relationship, $\cos(3x) = 4\cos^3(x) - 3\cos(x)$, is necessary for the proof.

$$\begin{aligned}
\cos(3x) &= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\
&= \cos(x)\left(\cos^2(x) - \sin^2(x)\right) - \sin(x)\left(2\sin(x)\cos(x)\right) \\
&= \cos(x)\left(\cos^2(x) - \left(1 - \cos^2(x)\right)\right) - 2\sin^2(x)\cos(x) \\
&= \cos(x)\left(2\cos^2(x) - 1\right) - 2\sin^2(x)\cos(x) \\
&= 2\cos^3(x) - \cos(x) - 2\left(1 - \cos^2(x)\right)\cos(x) \\
&= 2\cos^3(x) - \cos(x) - 2\cos(x) + 2\cos^2(x) \\
\cos(3x) &= 4\cos^3(x) - 3\cos(x)
\end{aligned}$$

Why is it impossible to construct the cosine of 20°?

$$\begin{aligned}
&\cos 60 = 1/2 \\
&\cos(3 \cdot 20) = 1/2
\end{aligned}$$

$$\begin{aligned}
&\cos(3 \cdot 20) = 4\cos^3(20) - 3\cos(20) \\
&4\cos^3 20 - 3\cos 20 = 1/2 \\
&4\cos^3 20 - 3\cos 20 - 1/2 = 0 \\
&8\cos^3 20 - 6\cos 20 - 1 = 0
\end{aligned}$$

if $x = \cos 20^\circ$, then $8x^3 - 6x - 1 = 0$.

We show that this cubic equation has no rational roots, and so cosine of 20° is not constructible.

The equation, $8x^3 - 6x - 1 = 0$, does not have any rational roots because using the Rational Roots Theorem, the only possible rational roots are $\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1$. None of these possibilities are roots. Therefore, the solutions to $8x^3 - 6x - 1 = 0$ are not constructible, so cosine of 20° cannot be constructed with a compass and straightedge alone. In conclusion, the 60° angle cannot be trisected with a compass and straightedge alone.

The level of some of the mathematics necessary to understand the proofs of the impossibility of the problems of antiquity is well above the high school level. However, most of the arguments can be understood by high school seniors. For example, students that have been exposed to trigonometric identities and the rational roots theorem could definitely analyze the mathematics that goes into showing these problems impossible. For students to fully comprehend what geometers of the past constructed, they need to take a step back in time.

At first, we would discuss the history of constructions and the tools allowed during the time of the Greeks. Initially, students will only be allowed paper, a compass, and a straightedge. Although students may get frustrated, it can be related to the difficulties that the mathematicians of the past might have had to overcome. This frustration is usually quickly forgotten once they transition to constructions using technology. Then they can see how the mathematicians of today have an easier task when it comes to analyzing constructions and their properties. Throughout the lesson, students will discuss the details about the constructions and why they are doing what they are doing. Students will complete the following constructions:

1. Construct a congruent segment.
2. Construct a congruent angle.
3. Construct an angle bisector.
4. Construct a line perpendicular to a given line at a point on the line.

5. Construct a line perpendicular to a given line from a given point not on the line.
6. Construct a perpendicular bisector of a line segment.
7. Construct a line parallel to a given line.
8. Construct an equilateral triangle.

Challenge Problem: Inscribe a circle in a triangle.

Challenge Problem: Circumscribe a circle about a triangle.

In relating back to the three impossible problems of antiquity, students will attempt to construct different angle measures. They will determine which ones can be constructed easily and why others cannot. This will just touch on the concept of constructible numbers, but will not get into the specific detail. While constructing these angles, students can also see the lengths that are created by constructing these angles. There can be some great discussion that can come from such constructions. Students will attempt the following angle constructions (not all are possible):

1. Construct a 90˚ angle.
2. Construct a 45˚ angle.
3. Construct a 60˚ angle.
4. Construct a 30˚ angle.
5. Construct a 120˚ angle.
6. Construct a 75˚ angle.
7. Construct a 20˚ angle.

Furthermore, students can mathematically analyze how the operations of addition, subtraction, multiplication, division, and square rooting can be shown through constructions. Therefore, students can show the construction of $x + y$ and $x - y$. Moreover, students can use the geometry they already know to show why the constructions for xy, y/x, and $\sqrt{x}$ work.

The importance of constructions can be argued for many reasons. According to Cathleen Sanders "construction can reinforce proof and lend visual clarity to many geometric relationships [18]." Robertson claims that constructions "give the secondary school student, starved for a Piagetian concrete-operational experience, something tangible [17].

In Pandisico's article *Alternative Geometric Constructions: Promoting Mathematical Reasoning* [16], he states that

> "…unless constructions simply ask students to mimic a given example, they promote true problem solving through the use of reasoning. Finally, constructions promote a spirit of exploration and discovery and can be guided to the extent that the teacher desires.

Overall, most mathematicians do agree that constructions are useful. However the weight that is given to compass and straightedge constructions is where there is a disagreement. Some educators use constructions as a topic to be covered only if time permits at the end of the year. Others think it should be incorporated throughout the course. It seems to depend on each individual's history in constructions as to which they prefer.

Most educators in general will agree there are several different types of student learning styles: auditory, visual, and manipulative. Constructions can be a nice combination of all of these. Due to age, physical or mental impairments, or just plain sloppiness, some students may have difficulties with the preciseness of compass and straightedge constructions. This lack of preciseness can sometimes be remedied with the use of technology.

Today, in the world of computer technology, there are many resources for a geometry classroom. The most common is "dynamic geometry" software. The word dynamic is used to describe the ability to "click and drag" the constructions to see that properties will always hold. Rather than having to construct the same type of construction again, this software allows for the construction to be "moved" to notice what will always hold true. The students will analyze the constructions and determine what changes and what stays the same. Students have the ability to manipulate shapes in order to investigate patterns, write conjectures, and test these conjectures [3]. On the other hand, the word used to describe compass and straightedge constructions is static. Static means immobile, stationary, unmoving, and fixed. Static constructions do not have the strong impact as those of dynamic constructions.

The first dynamic geometry software program was the *Geometric Supposer*. After advancements in technology the next program was *Cabri Géomètre.* This program was first incorporated onto the Texas Instruments graphing calculator, TI-92. *Cabri Junior* is now preloaded on the TI-84 Plus and the TI Voyage 200 and is also available for download on the TI-92 Plus, TI-89 Plus, and TI-83 Plus graphing calculators. The more commonly used program today is *The Geometer's*

Sketchpad. This program is preloaded on the TI Voyage 200 graphing calculator and is used primarily as computer software.

Kissane [6] describes how this type of software might change the outlook of geometry just as other technology has changed to focus in other areas of mathematics.

> "After the hand calculator was invented, arithmetic could never be the same again. Following the invention of data analysis software, statistics could never be the same again. Now that algebra is available not only on large computer systems, but also on graphics calculators and personal technologies like the TI-92, algebra and calculus can never be the same again. It now seems, too, that geometry can never be the same again. "

Not every mathematician is an advocate of this new advancement in geometry. Some believe it to be cheating. Some educators, in the past and even to this day, view the use of calculators as cheating in certain situations. But all educators realize the importance of student higher order thinking that can be achieved by analyzing problems instead of doing long arithmetic. In regards to constructions, the opponents of programs such as *The Geometer's Sketchpad* argue that students no longer realize the importance of proof. Students will mindlessly follow a set of directions and not pay attention to the details of what they are doing and why. Similarly, this could happen just as some students may mindlessly find the derivative of an equation and not know what they are in fact finding. This does not mean that students should not be taught the short cut of finding derivatives. There will be students that are not concerned about the why's of mathematics. But educators have to encourage an interest in mathematics as much as possible.

The use of technology in the geometry classroom has only been common in the past decade. Who knows how educators will view the use of dynamic geometry software in the future? Overall, however, most educators see the true benefits of using technology in the geometry classroom. As Brad Glass [3] answers the question of how technology can be used to help students learn geometry,

> "…computing tools can help students (a) focus on the relevant aspects of a problem or figure, (b) function at higher levels of geometric understanding, (c) distinguish between drawings and constructions, and (d) develop and reason about conjectures on the basis of generalizations of patterns that unfold during exploration."

Dynamic software programs help to visualize a relationship but do not provide a formal proof using appropriate geometry definition, postulates, and theorems. However, experimentation and analysis used with technology can be a great transition into formal proof.

There list of activities using dynamic geometry software as endless. Specifically, Key Curriculum Press has an *Exploring Geometry with The Geometer's Sketchpad* resource [1] that has ten chapters worth of materials. Teachers do not have to come up with activities on their own. The directions and diagrams are detailed and can be easily followed by students of varying abilities.

The topics covered are as follows…

1. Lines and Angles
2. Transformations, Symmetry, and Tessellations
3. Triangles
4. Quadrilaterals
5. Polygons
6. Circles
7. Area
8. The Pythagorean Theorem
9. Similarity
10. Trigonometry and Fractals

In addition to student activities, there is a multitude of guided demonstrations. These demonstrations can work well on an individual student basis and also as a teacher only demonstration.

Students can also use *The Geometer's Sketchpad* as an accurate compass and straightedge. For those students that are not accurate with a compass, *The Geometer's Sketchpad* can be an essential tool. They can understand the concepts and constructions without having to worry about human drawing inaccuracy. If students only use *The Geometer's Sketchpad* and do not discuss compass and straightedge construction, the true geometry behind these computer programs could be lost. Therefore, only after students have made some constructions using a compass and straightedge should students then make similar constructions using *The Geometer's Sketchpad*. Teachers have to decide how much of the program the students can use. Some teachers may only allow the point, line, and circle tools to be used. Whereas, other teachers may allow the shortcuts that *The Geometer's Sketchpad* offers. As the students construct using *The Geometer's Sketchpad* it is important to discuss how these dynamic constructions relate to compass and straightedge constructions and the history of constructions.

Regardless of how much a teacher uses technology or the method of integrating this technology, students and teachers alike enjoy the benefits of advancements in technology. The National Council of Teachers of Mathematics (NCTM) published *Principles and Standards for School Mathematics* in 2000 with specific geometry standards for grades 9-12 . In agreement with most students and teachers, this publication emphasized the importance of dynamic geometry software in the geometry classroom.

When deciding on curriculum, the state and national standards should be examined. The NCTM has a standard specific to Geometry. The broad standard is broken down into four sub-standards as follows:

1. Analyze characteristic and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
2. Specify locations and describe spatial relationships using coordinate geometry and other representational systems.
3. Apply transformations and use symmetry to analyze mathematical situations.
4. Use visualization, spatial reasoning, and geometric modeling to solve problems.

Although all of these standards are important, the ones that most directly deal with constructions are standards one and four. Standard four is further broken down into the following expectations:

- Draw and construct representations of two- and three-dimensional geometric objects using a variety of tools;
- Visualize three-dimensional objects and spaces from different perspectives and analyze their cross-sections;
- Use vertex-edge graphs to model and solve problems;
- Use geometric models to gain insights into, and answer question in, other areas of mathematics;
- Use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

Furthermore, an expectation of sub-standard one is to "establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others." These sub-standards and expectations clearly show that higher order geometric thinking is expected of high school students.

The most obvious expectation related to geometric constructions is that students should be able to "draw and construct representations of two- and three-dimensional geometric objects using a variety of tools." Clearly students can not ignore the constructions of the past. In order to use a variety of tools students may use a paper and pencil, compass and straightedge, or dynamic geometry software. Students should not focus on just one method but many.

Another standard set out by the NCTM is Reasoning and Proof. Students should be able to

- Recognize reasoning and proof as fundamental aspects of mathematics;
- Make and investigate mathematical conjectures;
- Develop and evaluate mathematical arguments and proofs;
- Select and use various types of reasoning methods of proof [9].

Again, students can not ignore formal proofs. Even though exploring and making conjectures using *The Geometer's Sketchpad* is helpful and interesting, it does not prove something to be true. This investigating and conjecturing can be a solid link to more formal geometry.

Illinois State Standards also have high expectations for high schoolers in terms of geometric thinking. Illinois' State Goal 9 states that student should "use geometric methods to analyze, categorize and draw conclusions about points, lines, planes, and space." This goal is further broken down as follows:

A. Demonstrate and apply geometric concepts involving points, lines, planes, and space.
B. Identify, describe, classify and compare relationships using points, lines, planes, and solids.
C. Construct convincing arguments and proofs to solve problems.
D. Use trigonometric ratios and circular functions to solve problems.

Similar to the NCTM standards, the Illinois standards are also broken down into specific expectations. Some of the related expectations are:

- 9.B.4 – Recognize and apply relationships within and among geometric figures.
- 9.C.4a – Construct and test logical arguments for geometric situations using technology where appropriate.
- 9.C.4b – Construct and communicate convincing arguments for geometric situations.
- 9.C.4c. – Develop and communicate mathematical proofs (e.g., two-column, paragraph, indirect) and counter examples for geometric statements [5].

Again, these standards clearly correlate with constructions both with and without the use of technology. Students must know how to think "geometrically." Constructions and proof are still a part of the state and national standards. Overall, geometry is a core part of mathematics. Students have had to know and will continue to have to know about constructions and proofs.

How students continue to learn about constructions and proof is changing. Some say for the better; some say for the worse. But with the advancement of technology and resources, students have more opportunities to see constructions and proofs in a variety of ways. However, just because the world is moving ahead does not mean that the history should be forgotten. The methods of today are just as important as the history of the past. With the knowledge of both the old and new methods, students can compare and contrast to determine which method better suites a particular problem. Just as the Oracle of Delos believed of the Athenians, teachers should believe of their students:

students need to recognize the importance of mathematics and geometry in general.

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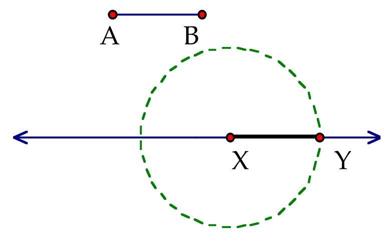
Constructions

1. Construct a congruent segment.

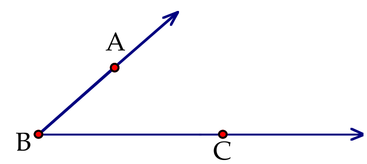
   Given: AB

   Construct: XY such that $AB = XY$



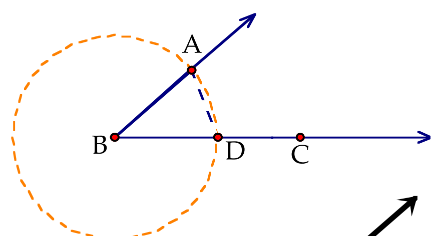
   1) Construct a line with X on the line.
   2) Construct a circle with a center at X and a radius of $|AB|$. Let Y be the point of intersection of circle and the line.
   3) Then $AB = XY$.

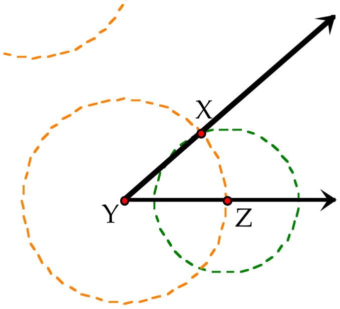
2. Construct a congruent angle.



   Given: $\angle ABC$

   Construct: $\angle XYZ$ such that $mABC = m\angle XYZ$



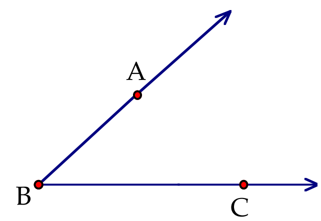


   1) Construct a line with Y on the line.
   2) Construct a circle with a center at Y and a radius of $|AB|$. Let Z be the point of intersection of circle and the line.
   3) Construct a circle with center at A and radius $|AB|$. Let D be the point of intersection of ray BC and this circle. Construct $\overline{AD}$.
   4) Construct a circle with a center at Z and a radius of $|AD|$. Let X be the point of intersection of the two circles.
   5) Construct $\overrightarrow{YX}$.
   6) Then $mABC = m\angle XYZ$.

3. Construct an angle bisector.

Given: $\angle ABC$

Construct: the angle bisector of $\angle ABC$

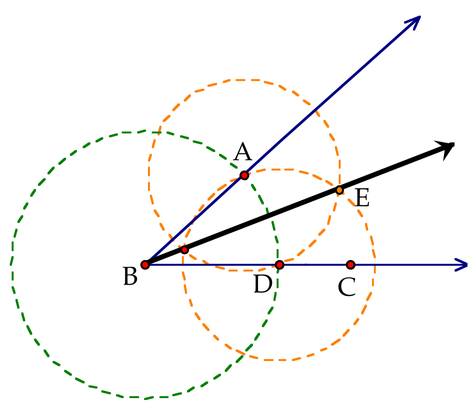


1) Construct a circle with a center at B and a radius of $|AB|$. Let D be the point of intersection of the circle and $\overrightarrow{BC}$.

2) Construct a circle with a center at A and a radius of $|AD|$. Construct a circle with a center at D and a radius of $|AD|$. Let E be a point of intersection of the two circles on the interior of $\angle ABC$.

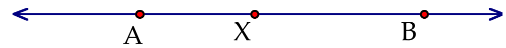
3) Construct $\overrightarrow{BE}$

4) Then $\overrightarrow{BE}$ is the angle bisector of $\angle ABC$.



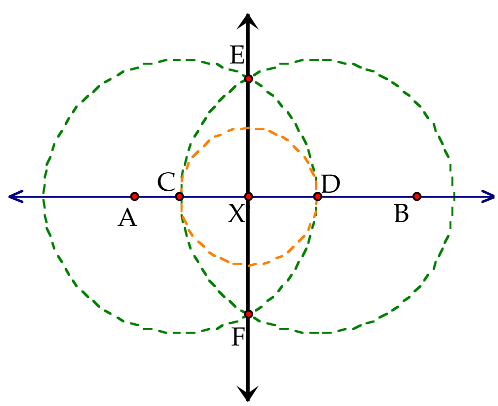
4. Construct a line perpendicular to a given line at a point on the line.

Given: a point X on $\overleftrightarrow{AB}$

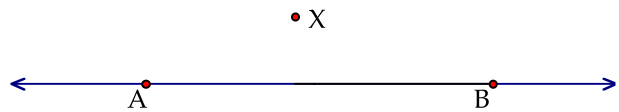


Construct: the a line perpendicular to $\overleftrightarrow{AB}$ through point X

1) Construct a circle with a center at X and a radius of $|XC|$, where C is on $\overleftrightarrow{AB}$. Let D be the other point of intersection of the circle and $\overleftrightarrow{AB}$.
2) Construct a circle with a center at C and a radius of $|CD|$. Construct a circle with a center at D and a radius of $|CD|$. Let E and F be the points of intersection of the two circles.
3) Construct $\overleftrightarrow{EF}$
4) Then $\overleftrightarrow{EF}$ is perpendicular to $\overleftrightarrow{AB}$ through point X.



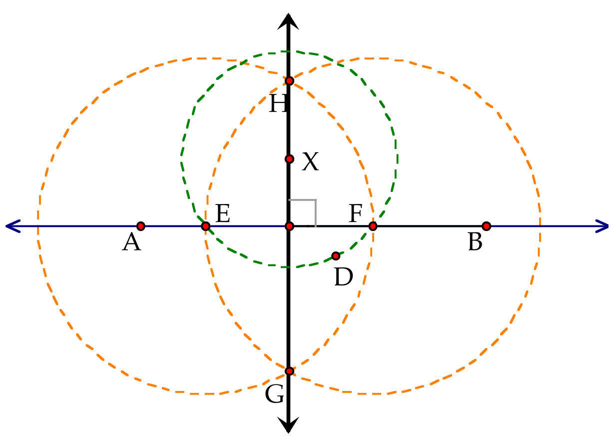
5. Construct a line perpendicular to a given line from a given point not on the line.



Given: a point X not on $\overleftrightarrow{AB}$

Construct: the a line perpendicular to $\overleftrightarrow{AB}$ through point X

1) Construct a circle with a center at X and a radius of $|XD|$, where $|XD|$ is greater than the distance from X to $\overleftrightarrow{AB}$. Let E and F be the points of intersection of the circle and $\overleftrightarrow{AB}$.
2) Construct a circle with a center at E and a radius of $|EF|$. Construct a circle with a center at F and a radius of $|EF|$. Let H and G be the points of intersection of the two circles.
3) Construct $\overleftrightarrow{HG}$
4) Then $\overleftrightarrow{HG}$ is perpendicular to $\overleftrightarrow{AB}$ through point X.



6. Construct a perpendicular bisector of a line segment.

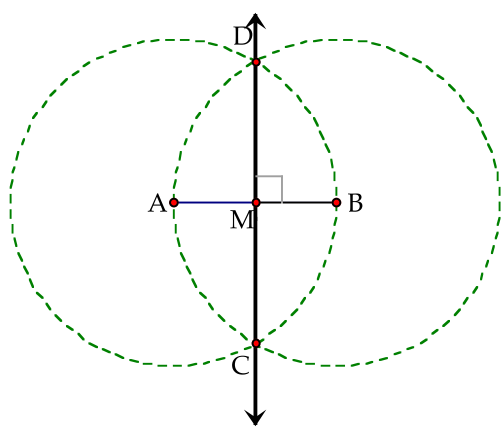
Given: $\overline{AB}$

Construct: the perpendicular bisector of $\overline{AB}$

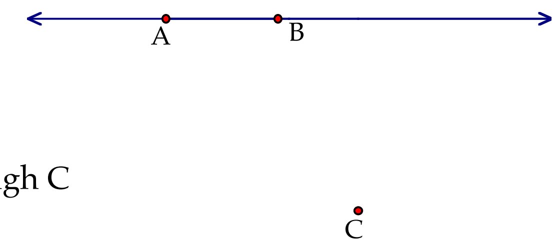
1) Construct a circle with a center at A and a radius of $|AB|$. Construct a circle with a center at B and a radius of $|AB|$. Let C and D be the points of intersection of the two circles.

2) Construct $\overleftrightarrow{CD}$

3) Then $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



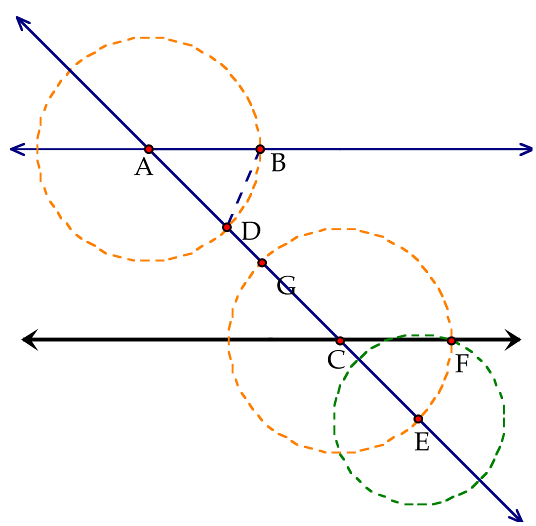
7. Construct a line parallel to a given line.



Given: $\overleftrightarrow{AB}$ and point C not on line $\overleftrightarrow{AB}$

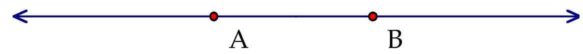
Construct: a line parallel to $\overleftrightarrow{AB}$ passing through C

1) Construct $\overleftrightarrow{AC}$.
2) Construct a circle with center at A and a radius of $|AB|$. Let D be the point of intersection of this circle with $\overrightarrow{AC}$.
3) Construct a circle with center at C and a radius of $|AB|$. Let E be the point of intersection, not between A and C, of this circle with $\overrightarrow{AC}$.
4) Construct a circle with center at E and a radius of $|DB|$. Let F be the point of intersection of this circle and circle C.
5) Construct $\overleftrightarrow{CF}$.
6) Then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CF}$.



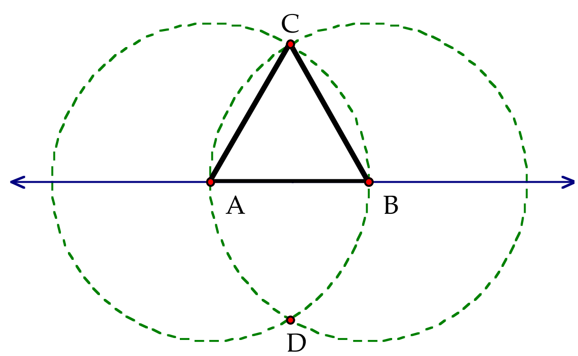
8. Construct an equilateral triangle.

Given: $\overleftrightarrow{AB}$



Construct: an equilateral triangle, ΔABC

1) Construct a circle with a center at A and a radius of $|AB|$. Construct a circle with a center at B and a radius of $|AB|$. Let C and D be the points of intersection of the two circles.
2) Construct ΔABC.
3) Then ΔABC is an equilateral triangle.



Challenge Problem: Inscribe a circle in a triangle.

Given: ΔABC

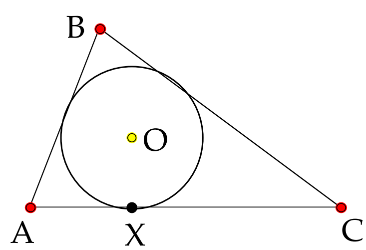
Construct: a circle inscribed in ΔABC

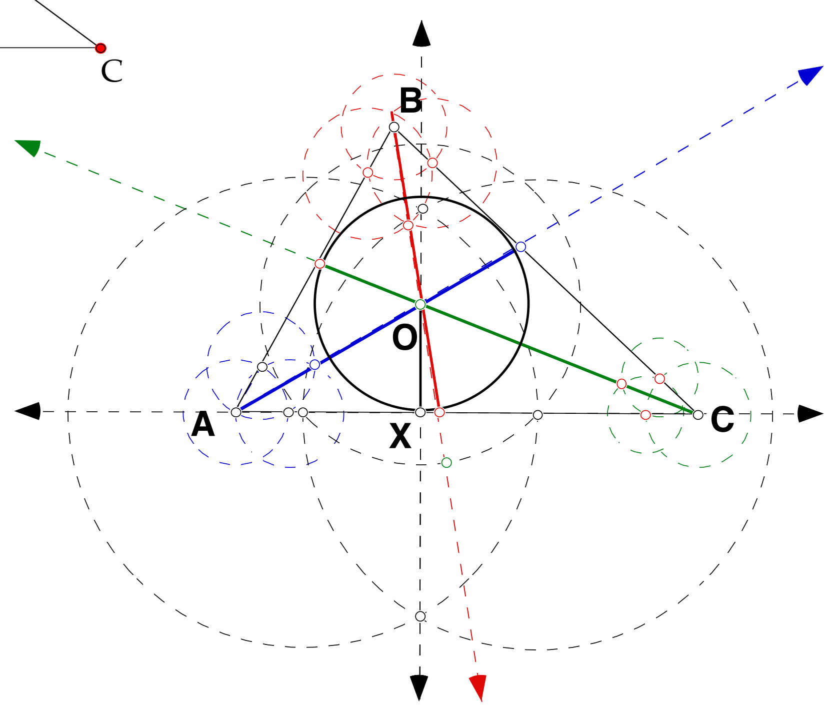
1) Construct the angle bisectors of the angles of ΔABC.

   (Only two of the three are really necessary to construct.)

2) Let O be the point of intersection of the angle bisectors.
3) Construct an altitude from O to any of the three sides.
4) Let X be the point of intersection of the altitude with the side of the triangle.
5) Construct a circle with center O and radius $|OX|$.
6) Then this circle is inscribed in the triangle ABC.

When inscribing a circle in a triangle, the angle bisectors will always meet inside the circle.





Challenge Problem: Circumscribe a circle about a triangle.

Given: ΔABC

Construct: a circle circumscribed about ΔABC

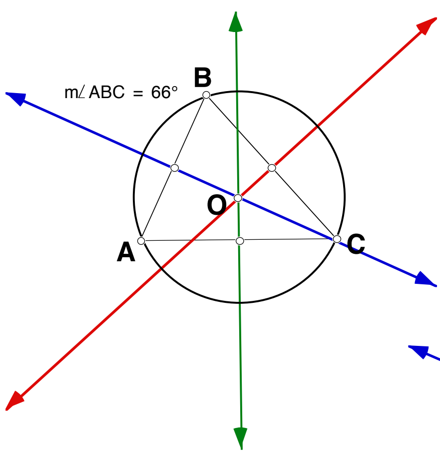
1) Construct $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$.
2) Construct the perpendicular bisectors of each side of the ΔABC.

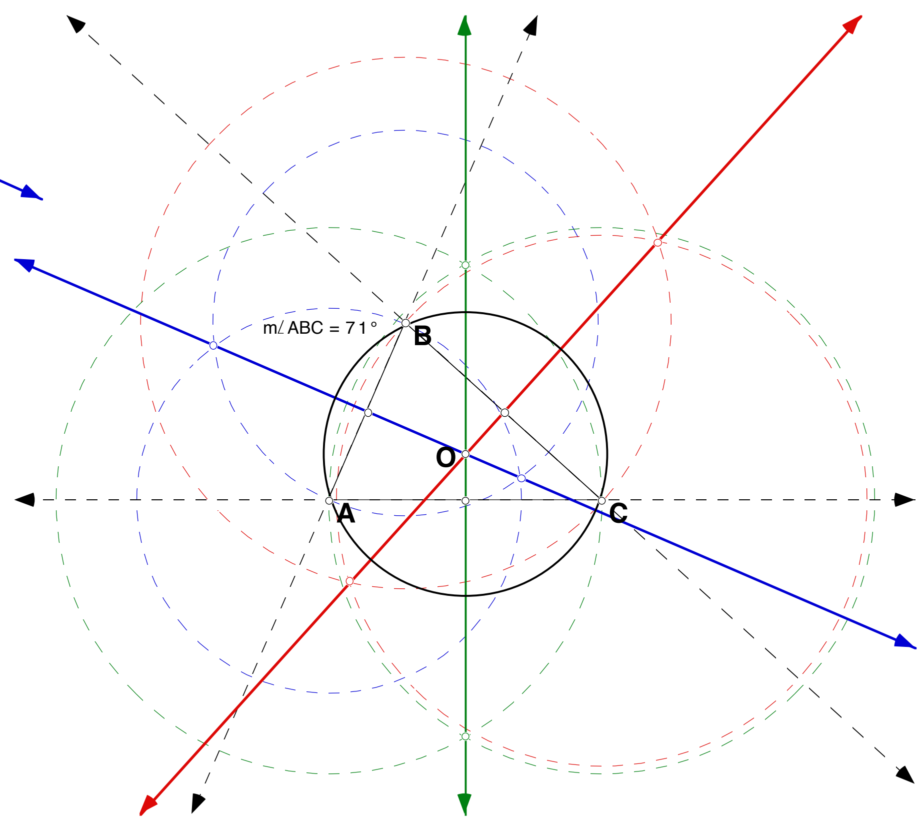
   (Only two of the three are really necessary to construct)
3) Let O be the point of intersection of the three perpendicular bisectors.

   Then point O is equidistant from points A, B, and C.
4) Construct a circle with center O and a radius of |OA|.
5) Then this circle is circumscribed about the triangle ABC.

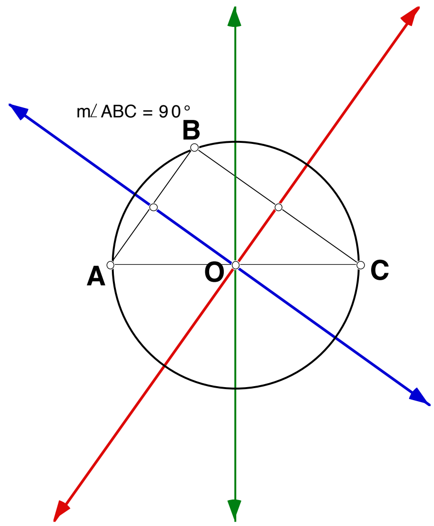
When circumscribing a circle, the perpendicular bisectors intersect in different places depending on the type of triangle.

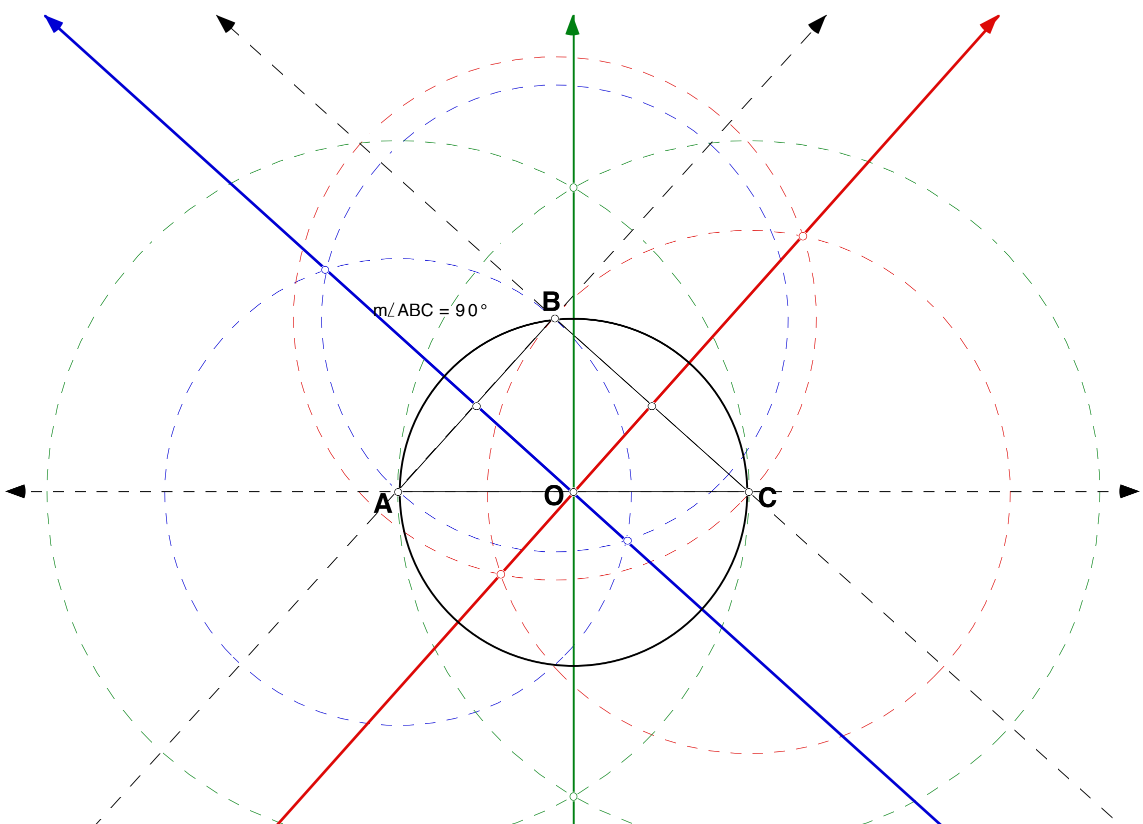
- In an acute triangle, the perpendicular bisectors intersect inside the triangle.



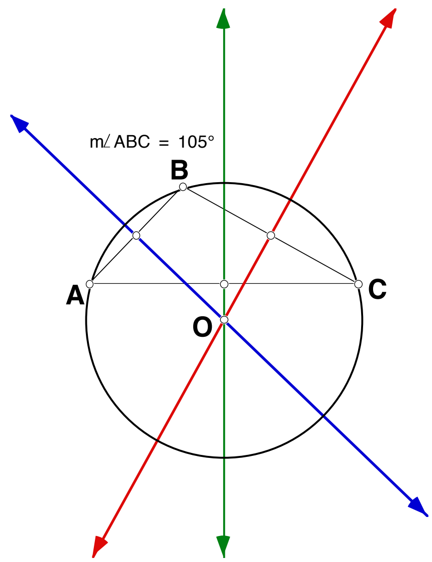


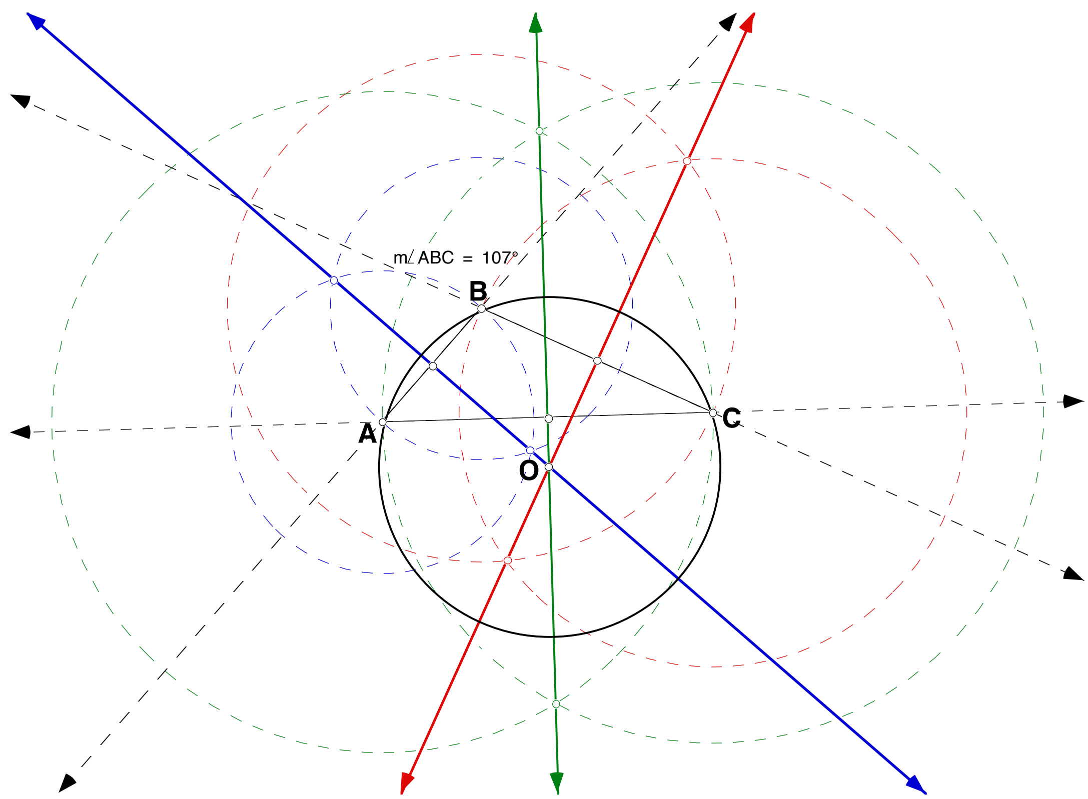
- In a right triangle, the perpendicular bisectors intersect on the triangle, specifically on the midpoint of the hypotenuse. The hypotenuse of the triangle is also the diameter of the circumscribed circle.





In an obtuse triangle, the perpendicular bisectors intersect outside the triangle.





1. Construct a 90˚ angle.

1) Construct $\overleftrightarrow{AB}$.

2) Construct a perpendicular bisector.

2. Construct a 45° angle.

   1) Construct $\overleftrightarrow{AB}$.

   2) Construct a perpendicular bisector.

   3) Construct the angle bisector.

3. Construct a 60° angle.

   1) Construct an equilateral triangle.

4. Construct a 30° angle.

   1) Construct an equilateral triangle.

   2) Construct the angle bisector.

5. Construct a 120° angle.

   1) Construct a 60° angle.

   2) Then construct 60° plus 60°.

6. Construct a 75° angle.

   1) Construct a 30° angle.

   2) Construct a 45° angle.

   3) Then construct a 30° plus 45°.

7. Construct a 20° angle. **NOT POSSIBLE!**