De Moivre's Theorem 10.4

Introduction

In this block we introduce De Moivre's theorem and examine some of its consequences. We shall see that one of its uses is in obtaining relationships between trigonometric functions of multiple angles (like $\sin 3x$, $\cos 7x$ etc) and powers of trigonometric functions (like $\sin^2 x$, $\cos^4 x$ etc). Another important aspect of De Moivre's theorem lies in its use in obtaining complex roots of polynomial equations. In this application we re-examine our definition of the argument $\arg(z)$ of a complex number.

Prerequisites

Before starting this Block you should ...

① be familiar with the polar form

② be familiar with the Argand diagram

③ be familiar with the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$

④ know how to expand $(x+y)^n$ when n is a positive integer

Learning Outcomes

After completing this Block you should be able to ...

✓ employ De Moivre's theorem in a number of applications

✓ understand more clearly the argument $\arg(z)$ of a complex number

✓ obtain complex roots of complex numbers

Learning Style

To achieve what is expected of you ...

☞ allocate sufficient study time

☞ briefly revise the prerequisite material

☞ attempt *every* guided exercise and most of the other exercises

1. De Moivre's Theorem

We have seen, in Block 10.3, that, in polar form, if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ then the product zw is easily obtained:

$$zw = rt(\cos(\theta + \phi) + \mathrm{i}\sin(\theta + \phi))$$

In particular, if $r = 1, t = 1$ and $\theta = \phi$ (i.e. $z = w = \cos\theta + \mathrm{i}\sin\theta$), we obtain

$$(\cos\theta + \mathrm{i}\sin\theta)^2 = \cos 2\theta + \mathrm{i}\sin 2\theta$$

Multiplying each side by $\cos\theta + \mathrm{i}\sin\theta$ gives

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = (\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos\theta + \mathrm{i}\sin\theta) = (\cos 3\theta + \mathrm{i}\sin 3\theta)$$

on adding the arguments of the terms in the product.
Similarly

$$(\cos\theta + \mathrm{i}\sin\theta)^4 = (\cos 4\theta + \mathrm{i}\sin 4\theta).$$

After completing n such products we have:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

where n is a positive integer.
In fact this result can be shown to be true for those cases in which n is a negative integer and even when n is a rational number e.g. $n = \frac{1}{2}$.

Key Point

If p is a rational number:

$$(\cos\theta + \mathrm{i}\sin\theta)^p = \cos p\theta + \mathrm{i}\sin p\theta$$

This result is known as **De Moivre's Theorem**.

In exponential form De Moivre's theorem, in the case when p is a positive integer, is simply a statement of the laws of indices:

$$(\mathrm{e}^{\mathrm{i}\theta})^p = \mathrm{e}^{\mathrm{i}p\theta}$$

Example Use De Moivre's theorem to obtain an expression for $\cos 3\theta$ in terms of powers of $\cos\theta$ alone.

Solution

From De Moivre's theorem (Key Point above with $p = 3$) we have

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$$

However, expanding the left-hand side (using: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$) we have:

$$\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta = \cos 3\theta + \mathrm{i}\sin 3\theta$$

and then, equating the real parts of both sides, gives the relation:

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$$

or, replacing $\sin^2\theta$ by $(1-\cos^2\theta)$;

$$\cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos 3\theta$$

Finally:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

is the required relation.

Now do this exercise

Use the last example to find an expression for $\sin 3\theta$ in terms of powers of $\sin\theta$ alone.

Answer

Now do this exercise

Without using tables or a calculator obtain the value of $\sin 60^0$ given that $\sin 20^0 \approx 0.342020$

Answer

2. De Moivre's Theorem and Root Finding

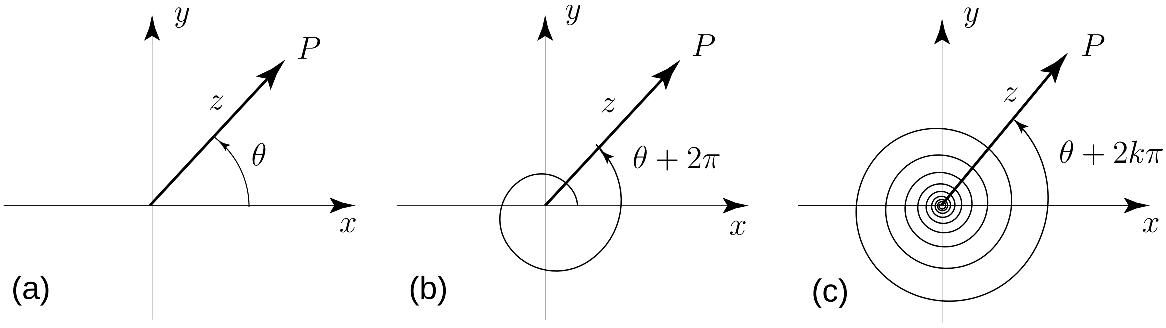
In this section we ask if we can obtain fractional powers of complex numbers; for example what are the values of $8^{1/3}$ or $(-24)^{1/4}$ or even $(1+\mathrm{i})^{1/2}$?
More precisely, for these three examples, we are asking for those values of z which satisfy

$$z^3 - 8 = 0 \qquad \text{or} \qquad z^4 + 24 = 0 \qquad \text{or} \qquad z^2 - (1+\mathrm{i}) = 0$$

Each of these problems involve finding roots of a complex number.
To solve problems such as these we shall need to be more careful with our interpretation of $\arg(z)$ for a given complex number z.

Arg(z) revisited

By definition $\arg(z)$ is the angle made by the line representing z with the positive x-axis. See (a) in the following diagram. However, as the second diagram (b) shows you can increase θ by 2π (or 360^0) and still obtain the *same* line in the xy plane. In general, as indicated in diagram (c) any integer multiple of 2π can be added to or subtracted from $\arg(z)$ without affecting the Cartesian form of the complex number.



Key Point

$\arg(z)$ is unique only up to an integer multiple of 2π

For example: we have previously written

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}) \qquad \text{in polar form}$$

However, we could also write, equivalently:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2\pi))$$

or, in full generality:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \qquad k = 0, \pm 1, \pm 2, \ldots$$

This last expression shows that in the polar form of a complex number the argument of z, $\arg(z)$ can assume many different values, each one differing by an integer multiple of 2π. This is nothing more than a consequence of the well-known properties of the trigonometric functions:

$$\cos(\theta + 2k\pi) \equiv \cos\theta, \qquad \sin(\theta + 2k\pi) \equiv \sin\theta \qquad \text{for } \textit{any} \text{ integer } k$$

We shall now show how we can use this more general interpretation of $\arg(z)$ in the process of finding roots.

Example Find all the values of $8^{1/3}$.

Solution

Solving $z = 8^{1/3}$ for z is equivalent to solving the cubic equation $z^3 - 8 = 0$. We expect that there are three possible values of z satisfying this cubic equation. Thus, rearranging: $z^3 = 8$. Now write the right-hand side as a complex number in polar form:

$$z^3 = 8(\cos 0 + \mathrm{i}\sin 0)$$

(i.e. $r = |8| = 8$ and $\arg(8) = 0$). However, if we now generalise our expression for the argument, by adding an arbitrary integer multiple of 2π, we obtain the modified expression:

$$z^3 = 8(\cos(2k\pi) + \mathrm{i}\sin(2k\pi)) \qquad k = 0, \pm 1, \pm 2, \ldots$$

Now take the cube root of both sides

$$\begin{aligned} z &= \sqrt[3]{8}(\cos(2k\pi) + \mathrm{i}\sin(2k\pi))^{\frac{1}{3}} \\ &= \sqrt[3]{8}(\cos\frac{2k\pi}{3} + \mathrm{i}\sin\frac{2k\pi}{3}) \qquad \text{using De Moivre's theorem} \end{aligned}$$

Now in this expression k can take any integer value or zero. The normal procedure is to take three consecutive values of k (say $k = 0, 1, 2$). Any other value of k chosen will lead to a root (a value of z) which repeats one of the three already determined.

$$\begin{aligned} \text{So if} \quad k &= 0 \qquad z_0 = 2(\cos 0 + \mathrm{i}\sin 0) = 2 \\ k &= 1 \qquad z_1 = 2(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}) = -1 + \mathrm{i}\sqrt{3} \\ k &= 2 \qquad z_2 = 2(\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}) = -1 - \mathrm{i}\sqrt{3} \end{aligned}$$

These are the three (complex) values of $8^{\frac{1}{3}}$. The reader should verify, by direct multiplication, that $(-1+\mathrm{i}\sqrt{3})^3 = 8$ and that $(-1-\mathrm{i}\sqrt{3})^3 = 8$. The reader may have noticed within this example a subtle change in notation. When we write, for example, $8^{1/3}$ then we are expecting three possible values, as calculated above. However, when we write $\sqrt[3]{8}$ then we are only expecting one value; that delivered by your calculator.

In the above example we have worked with the polar form. Precisely the same calculation can be carried through using the exponential form of a complex number. We take this opportunity to repeat this calculation but working exclusively in exponential form.

Thus

$$\begin{aligned} z^3 &= 8 \\ &= 8\mathrm{e}^{\mathrm{i}(0)} \qquad (\text{i.e. } r = |8| = 8 \quad \text{and} \quad \arg(8) = 0) \\ &= 8\mathrm{e}^{\mathrm{i}(2k\pi)} \qquad k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

therefore taking cube roots

$$\begin{aligned} z &= \sqrt[3]{8}\left[\mathrm{e}^{\mathrm{i}(2k\pi)}\right]^{\frac{1}{3}} \\ &= \sqrt[3]{8}\mathrm{e}^{\frac{\mathrm{i}2k\pi}{3}} \qquad \text{using De Moivre's theorem} \end{aligned}$$

Again k can take any integer value or zero. Any three consecutive values will give the roots.

$$\begin{array}{llll} \text{So if} & k = 0 & & z_0 = 2e^{i0} = 2 \\ & k = 1 & & z_1 = 2e^{\frac{i2\pi}{3}} = -1 + i\sqrt{3} \\ & k = 2 & & z_2 = 2e^{\frac{i4\pi}{3}} = -1 - i\sqrt{3} \end{array}$$

These are the three (complex) values of $8^{\frac{1}{3}}$ obtained using the exponential form. Of course at the end of the calculation we have converted back to standard Cartesian form.

Try each part of this exercise

Following the procedure outlined in the previous example obtain the two complex values of $(1 + i)^{1/2}$.

Part (a) Begin by obtaining the polar form (using the general form of the argument) of $(1 + i)$.

Answer

Part (b) Now take the square root and use De Moivre's theorem to complete the solution.

Answer

More exercises for you to try

1. Use De Moivre's theorem to obtain expansions for $\cos 2\theta$ and $\sin 2\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.
2. Without using tables or a calculator find an expression for $\cos 30^0$ given only that $\cos 90^0 = 0$.
3. Find all those values of z which satisfy $z^4 + 1 = 0$. Write your values in standard Cartesian form.
4. Obtain the real and imaginary parts of $\sinh(1 + \frac{i\pi}{6})$

Answer

3. Computer Exercise or Activity

For this exercise it will be necessary for you to access the computer package DERIVE.
In DERIVE the basic complex object i is denoted by $\hat{\imath}$. You can use this in any expression by keying $\boxed{\text{ctrl}}+i$ or by clicking on the $\hat{\imath}$ icon in the Expression dialog box. The conjugate of a complex number z is written conj(z) in DERIVE and the modulus of z is written abs(z). DERIVE will help you verify your complex number solutions to the Block exercises.

DERIVE will be able to confirm many of the expansions for $\cos n\theta$ and $\sin n\theta$ used in this block. However, you will first need to key Declare:Algebra-State:Simplification
Then go into the Trigonometry dialog box and choose Expand then DERIVE responds with: Trigonometry:=Expand As an example if we wish to find the expansion of $\cos 3x$ we would key in this expression in the usual wayfollowed by Simplify:Basic DERIVE responds:

$\text{COS}(x) \cdot (1 - 4 \cdot \text{SIN}(x)^2)$

Similarly DERIVE can expand exponential complex numbers into polar form. This time you would key in: Declare:Algebra-State:Simplification but this time choose Expand from the Exponential dialog box. DERIVE will respond with Exponential:=Expand For example if you input $\mathrm{e}^{\mathrm{i}\theta}$ by keying $\hat{e} \wedge (\hat{\imath}\theta)$ in the Expression dialog box and then choose Simplify:Basic, DERIVE responds with

$\text{COS}(\theta) + \hat{\imath} \cdot \text{SIN}(\theta)$

DERIVE will also find (some) complex roots. For example if you key in $(1 + \hat{\imath}) \wedge (0.5)$ followed by Simplify:Basic DERIVE responds with

$$\sqrt{\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)} + \hat{\imath} \cdot \sqrt{\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)}$$

This can be written in numerical form using Simplify:Approximate. However, useful though this is, it nevertheless only gives *one* of the required roots.

End of Block 10.4

You should obtain $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ since, from the previous example (but this time equating imaginary parts of both sides)

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Back to the theory

You should obtain $\sin 60^0 \approx 0.866025$ since, from the previous example, choosing $\theta = 20^0$ we obtain:

$$\sin 60^0 = 3\sin 20^0 - 4\sin^3 20^0 \approx 0.866025 \qquad (= \frac{\sqrt{3}}{2} \quad \text{exactly})$$

Back to the theory

You should obtain $1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \quad k = 0, \pm 1, \pm 2, + \ldots.$

Back to the theory

You should obtain

$$z_1 = \sqrt[4]{2}(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}) = 1.099 + 0.455\mathrm{i}$$
$$z_2 = \sqrt[4]{2}(\cos(\frac{\pi}{8} + \pi) + \mathrm{i}\sin(\frac{\pi}{8} + \pi)) = -1.099 - 0.455\mathrm{i}$$

A good exercise would be to repeat the calculation using the exponential form.

Back to the theory

1. $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\cos\theta\sin\theta$
2. $\cos 30^0 = \dfrac{\sqrt{3}}{2}$
3. $z_0 = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} \qquad z_1 = -\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} \qquad z_2 = -\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}} \qquad z_3 = \dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$
4. $\sinh(1 + \dfrac{i\pi}{6}) = \dfrac{\sqrt{3}}{2}\sinh 1 + \dfrac{i}{2}\cosh 1 = 1.0178 + i(0.7715)$

Back to the theory