BINOMIAL THEOREM

1. Introduction

When we expand $(x+1)^2$ and $(x+1)^3$, we get

$$\begin{aligned}(x+1)^2 &= (x+1)(x+1) \\ &= x^2 + x + x + 1 \\ &= x^2 + 2x + 1\end{aligned}$$

and

$$\begin{aligned}(x+1)^3 &= (x+1)(x+1)^2 \\ &= (x+1)(x^2 + 2x + 1) \\ &= x^3 + 2x^2 + x + x^2 + 2x + 1 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

respectively. However, when we try to expand $(x+1)^4$ and $(x+1)^5$, we find that this method becomes very tedious. Furthermore, how about $(x+1)^n$, when n is an arbitrary positive integer? The binomial theorem helps us to expand these and other expressions more easily and conveniently.

Here we shall go over the basics of the binomial theorem, as well as a number of techniques in which the binomial theorem helps us to solve a wide range of problems.

2. The Binomial Coefficients

Definition 2.1. (Binomial coefficient)

The **binomial coefficients** are the coefficients in the expansion of $(x+1)^n$. The coefficient of x^r in $(x+1)^n$ is denoted as C_r^n, ${}_nC_r$ or $\binom{n}{r}$.

(Here we shall be using the first notation throughout.)

The **Pascal triangle** lists out all the binomial coefficients, as shown in Figure 1.

0th row	1							
1st row	1	1						
2nd row	1	2	1					
3rd row	1	3	3	1				
4th row	1	4	6	4	1			
5th row	1	5	10	10	5	1		
6th row	1	6	15	20	15	6	1	
7th row	1	7	21	35	35	21	7	1

Figure 1: The Pascal triangle

The 0th row is the coefficient in the expansion of $(x+1)^0$, or simply 1. From the first row onwards, the *n*th row contains the $n+1$ coefficients in the expansion of $(x+1)^n$. As you can probably observe, there are some intriguing properties in the Pascal triangle:

- The triangle is symmetric about the middle of each row, i.e. $C_r^n = C_{n-r}^n$.

- Except for the leftmost and rightmost numbers in each row, every number is the sum of the two numbers directly above it, i.e. $C_r^n = C_{r-1}^{n-1} + C_r^{n-1}$.

- In every row, the numbers first increase from 1 to a maximum value, then decrease back to 1.

- The second number (as well as the second last number) in the *n*th row is *n*, i.e. $C_1^n = C_{n-1}^n = n$.

Example 2.1.

Using the Pascal triangle, expand $(x+1)^5$ and $(x+1)^7$.

Solution.

From the Pascal triangle, the terms in the 5th row are 1, 5, 10, 10, 10, 5 and 1.

Thus we have $(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$.

Similarly, by referring to the 7th row of the Pascal triangle, we have

$$(x+1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1.$$

3. The Binomial Theorem

We are going to find an explicit formula for computing the numbers in the Pascal triangle, so that we can carry out expansions conveniently without always referring to the triangle as well as explain the interesting properties we observed.

According to definition,

$$(x+1)^n = C_n^n x^n + C_{n-1}^n x^{n-1} + C_{n-2}^n x^{n-2} + \cdots + C_2^n x^2 + C_1^n x + C_0^n \qquad (1)$$

Putting $x=0$, we get $C_0^n = 1$, as we can see from the Pascal triangle. The other terms can also be computed by successive differentiation of the above formula, as we illustrate below.

Differentiating (1) once, we have

$$n(x+1)^{n-1} = nC_n^n x^{n-1} + (n-1)C_{n-1}^n x^{n-2} + (n-2)C_{n-2}^n x^{n-3} + \cdots + 2C_2^n x + C_1^n \,.$$

Putting $x=0$ again, we have $C_1^n = n$.

Similarly, by differentiating (1) r times, we have

$$\begin{aligned} &n(n-1)(n-2)\cdots(n-r+1)(x+1)^{n-r} \\ &\quad = n(n-1)(n-2)\cdots(n-r+1)C_n^n x^{n-r} + \cdots + (r+1)r\cdots 3\cdot 2C_{r+1}^n x + r\cdots 3\cdot 2\cdot 1C_r^n \end{aligned}$$

Putting $x=0$, we get $C_r^n = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r\cdots 3\cdot 2\cdot 1}$.

To make the formula simpler, we introduce the **factorial** symbol.

Definition 3.1. (Factorial)

For positive integer *n*, we define the **factorial** of *n*, denoted as *n*!, by

$$n! = n\times(n-1)\times(n-2)\times\cdots\times 3\times 2\times 1.$$

We also define $0! = 1$.

With this notation in hand, we can give the formula for the binomial coefficients in a nice form:

Theorem 3.1.

The binomial coefficient C_r^n is given by $C_r^n = \dfrac{n!}{r!(n-r)!}$.

- In combinatorics, C_r^n denotes the number of ways of choosing *r* objects from *n* objects. The coefficient of x^r in $(x+1)^n$ is C_r^n and this has its combinatorial meaning. The term x^r comes by choosing *r* '*x*'s from *n* terms of $(x+1)$. Thus there are C_r^n terms and hence its coefficient is C_r^n.

Although we have till now only discussed the expansion of $(x+1)^n$, but for the general expression $(a+b)^n$ it just takes a few more steps.

Theorem 3.2. (Binomial Theorem)

$(a+b)^n = C_n^n a^n + C_{n-1}^n a^{n-1}b + C_{n-2}^n a^{n-2}b^2 + \cdots + C_1^n ab^{n-1} + C_0^n b^n$

$(a-b)^n = C_n^n a^n - C_{n-1}^n a^{n-1}b + C_{n-2}^n a^{n-2}b^2 - \cdots + (-1)^{n-1} C_1^n ab^{n-1} + (-1)^n C_0^n b^n$

Proof: We have

$$(a+b)^n = b^n\left(\frac{a}{b}+1\right)^n$$

and so

$$\left(\frac{a}{b}+1\right)^n = C_n^n\left(\frac{a}{b}\right)^n + C_{n-1}^n\left(\frac{a}{b}\right)^{n-1} + C_{n-2}^n\left(\frac{a}{b}\right)^{n-2} + \cdots + C_2^n\left(\frac{a}{b}\right)^2 + C_1^n\left(\frac{a}{b}\right) + C_0^n .$$

Consequently,

$$\begin{aligned}
&(a+b)^n \\
&= b^n\left[C_n^n\left(\frac{a}{b}\right)^n + C_{n-1}^n\left(\frac{a}{b}\right)^{n-1} + C_{n-2}^n\left(\frac{a}{b}\right)^{n-2} + \cdots + C_2^n\left(\frac{a}{b}\right)^2 + C_1^n\left(\frac{a}{b}\right) + C_0^n\right] \\
&= b^n\left[C_n^n a^n b^{-n} + C_{n-1}^n a^{n-1}b^{1-n} + C_{n-2}^n a^{n-2}b^{2-n} + \cdots + C_2^n a^2 b^{-2} + C_1^n ab^{-1} + C_0^n\right] \\
&= C_n^n a^n + C_{n-1}^n a^{n-1}b + C_{n-2}^n a^{n-2}b^2 + \cdots + C_2^n a^2 b^{n-2} + C_1^n ab^{n-1} + C_0^n b^n
\end{aligned}$$

as desired. For the expansion of $(a-b)^n$, we simply apply the above result with *b* replaced by –*b*.

Q.E.D.

Example 3.1.

Expand $(x-2)^{10}$ in ascending powers of *x*, up to the term containing x^3.

Solution.

$$\begin{aligned}(x-2)^{10} &= (-2+x)^{10} \\ &= C_0^{10}(-2)^{10} + C_1^{10}(-2)^9 x + C_2^{10}(-2)^8 x^2 + C_3^{10}(-2)^7 x^3 + \cdots \\ &= 1\cdot 1024 + 10\cdot(-512)x + \frac{10\cdot 9}{2\cdot 1}\cdot 258x^2 + \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}(-128)x^3 + \cdots \\ &= 1024 - 5120x + 11610x^2 - 15360x^3 + \cdots\end{aligned}$$

Example 3.2.

Find the coefficient of x^3 in $(1+x+2x^2)(1-2x)^5$.

Solution.

Since

$$\begin{aligned}(1-2x)^5 &= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + \cdots \\ &= 1 - 10x + 40x^2 - 80x^3 + \cdots\end{aligned}$$

the x^3 term in $(1+x+2x^2)(1-2x)^5$ is given by the sum of $1(-80x^3)$, $x(40x^2)$ and $2x^2(-10x)$. So the coefficient of x^3 is $-80+40-20=-60$.

Example 3.3.

Find the constant term in the expansion of $(x^2 - \frac{2}{x})^6$.

Solution.

In the expansion of $(x^2 - \frac{2}{x})^6$, in descending powers of *x*, the $(r+1)$ st term is

$$\begin{aligned}C_r^6 (x^2)^{6-r}\left(-\frac{2}{x}\right)^r &= C_r^6 x^{12-2r}\cdot(-2)^r x^{-r} \\ &= C_r^6 (-2)^r x^{12-3r}\end{aligned}$$

For the constant term, the power of *x* is 0. Thus we have 12 – 3*r* = 0 and thus *r* = 4.

Then the constant term is $C_4^6(-2)^4 = 15\cdot 16 = 240$.

Example 3.4.

Prove that $C_{a+1}^{n+1} \geq 2C_a^n \sqrt{\dfrac{n-a}{a+1}}$.

Solution.

The inequality is equivalent to

$$\frac{(n+1)!}{(a+1)!\,(n-a)!} \geq 2\frac{n!}{a!\,(n-a)!}\sqrt{\frac{n-a}{a+1}} .$$

After simplification, this becomes

$$\frac{n+1}{2} \geq \sqrt{(n-a)(a+1)} .$$

Using the AM-GM inequality, we have

$$\frac{n+1}{2} = \frac{(n-a)+(a+1)}{2} \geq \sqrt{(n-a)(a+1)}$$

and the original inequality is proved.

Example 3.5.

Find the maximum coefficient in the expansion of $(3x+5)^{10}$ without actual expansion.

Solution.

Note that the *r*-th term in the expansion is $C_r^{10}(3x)^r(5)^{10-r}$.

We consider the ratio of the $(r+1)$ st term to the *r*-th term, i.e.

$$\frac{C_{r+1}^{10}3^{r+1}\cdot 5^{9-r}}{C_r^{10}3^r\cdot 5^{10-r}} = \frac{3}{5}\cdot\frac{10-r}{r+1} .$$

To see which coefficient is maximum, we only need to know when the ratio is greater than 1 and when it is smaller than 1. So we set

$$\frac{3}{5}\cdot\frac{10-r}{r+1} > 1$$

$$r > \frac{25}{8} = 3\frac{1}{8}$$

Therefore, the coefficients increase at first and reach a maximum when $r = 4$.

So, the maximum coefficient is $C_4^{10}3^4\cdot 5^6$ = 265781250.

4. Applications of the Binomial Theorem

The obvious 'application' of the binomial theorem is to help us to expand algebraic expressions easily and conveniently. Other than that, the binomial theorem also helps us with simple numerical estimations. We illustrate this with a couple of examples.

Example 4.1.

A man put $1000 into a bank at an interest rate of 12% per annum, compounded monthly. How much interest can he get in 5 months, correct to the nearest dollar?

Solution.

Of course with a calculator in hand this will be of no difficulty. But without it we can still estimate the answer using the binomial theorem.

After 6 months, the amount will be $1000\times(1.01)^5$. Using the binomial theorem,

$$\begin{aligned}(1.01)^5 &= (1+0.01)^5\\ &= 1+5(0.01)+10(0.01)^2+10(0.01)^3+5(0.01)^4+(0.01)^5\end{aligned}$$

Since we only require the answer to be accurate to the nearest dollar, we take only the first three terms and ignore the rest.

We have $(1.01)^5 \approx 1+0.05+0.001=1.051$.

Thus the amount is approximately $1051 and the interest is approximately $51.

Example 4.2.

Estimate the value of $(1.0309)^6$ correct to 3 decimal places.

Solution.

Again with a calculator this will just take a few seconds.

But without it let us try to expand $(1+x+x^2)^6$.

We have $$\begin{aligned}(1+x+x^2)^6 &= \left[1+x(1+x)\right]^6\\ &= 1+6x(1+x)+15x^2(1+x)^2+20x^3(1+x^3)+\cdots\\ &= 1+6x+21x^2+50x^3+\cdots\end{aligned}$$

Putting $x=0.03$, we get $(1.0309)^6 \approx 1.200$.

5. Techniques for Dealing with Binomial Coefficients

Although we defined the binomial coefficients as coefficients in the binomial expansion of $(x+1)^n$, we have remarked that these coefficients have a very nice combinatorial interpretation and the coefficients themselves have many nice properties. In this section we shall demonstrate various techniques in working with binomial coefficients.

In Section 3, we worked out an explicit formula for the binomial coefficients by differentiation and substitution of the variable by certain numbers. It turns out that these are important techniques that can help us to derive other properties concerning the binomial coefficients as well.

Recall that, according to our definition,

$$(x+1)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n . \tag{2}$$

By putting $x=1$, we get

$$2^n = C_0^n + C_1^n + C_2^n + \cdots + C_n^n .$$

Similarly, by putting $x=-1$, we get

$$0 = C_0^n - C_1^n + C_2^n - \cdots + (-1)^n C_n^n .$$

Adding and subtracting these two equalities respectively and dividing by 2, we get

$$C_0^n + C_2^n + C_4^n + \cdots = C_1^n + C_3^n + C_5^n + \cdots = 2^{n-1} .$$

Now integrating (2) (we use definite integral here so that we can get rid of the constant of integration), we have

$$\int_0^x (1+x)^n dx = \int_0^x \left(C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n \right) dx$$
$$\frac{(1+x)^{n+1}-1}{n+1} = \frac{C_0^n}{1} x + \frac{C_1^n}{2} x^2 + \frac{C_2^n}{3} x^3 + \cdots + \frac{C_n^n}{n+1} x^{n+1}$$

Putting $x=1$, we have

$$\frac{2^{n+1}-1}{n+1} = \frac{C_0^n}{1} + \frac{C_1^n}{2} + \frac{C_2^n}{3} + \cdots + \frac{C_n^n}{n+1} .$$

Integrating once more, we have

$$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{C_0^n}{1\cdot 2} x^2 + \frac{C_1^n}{2\cdot 3} x^3 + \frac{C_2^n}{3\cdot 4} x^4 + \cdots + \frac{C_n^n}{(n+1)(n+2)} x^{n+2} .$$

Putting $x=1$ again, we see that

$$\frac{1}{n+2}=\frac{C_0^n}{1\cdot 2}-\frac{C_1^n}{2\cdot 3}+\frac{C_2^n}{3\cdot 4}-\cdots+(-1)^n\frac{C_n^n}{(n+1)(n+2)}.$$

Now consider the identity $(1+x)^m(1+x)^n\equiv(1+x)^{m+n}$. We shall compare the coefficient of x^r (assuming $0\le m\le n\le r$) on both sides. On the left hand side, we have

$$(1+x)^m(1+x)^n=\left(C_0^m+C_1^m x+C_2^m x^2+\cdots\right)\left(C_0^n+C_1^n x+C_2^n x^2+\cdots\right)$$

so that the coefficient of x^r is

$$C_0^m C_r^n+C_1^m C_{r-1}^n+C_2^m C_{r-2}^n+\cdots+C_r^m C_0^n.$$

On the right hand side, the coefficient of x^r is simply C_r^{m+n}. So we have the equality

$$C_0^m C_r^n+C_1^m C_{r-1}^n+C_2^m C_{r-2}^n+\cdots+C_r^m C_0^n=C_r^{m+n}. \tag{3}$$

Note that the identity $C_{r-1}^m+C_r^m=C_r^{m+1}$ (which we stated in Section 2 without proof) can be obtained using the above method by setting $n=1$. It can also be proved by applying the formula for binomial coefficients directly, and we leave it as an exercise.

Now setting $r=m=n$ in (3), we have

$$C_0^n C_n^n+C_1^n C_{n-1}^n+C_2^n C_{n-2}^n+\cdots+C_n^n C_0^n=C_n^{2n}.$$

Since $C_r^n=C_{n-r}^n$, this can also be expressed as

$$\left(C_0^n\right)^2+\left(C_1^n\right)^2+\left(C_2^n\right)^2+\cdots+\left(C_n^n\right)^2=C_n^{2n}.$$

So far we have seen some identities involving the binomial coefficients. We include below a few more examples which establish further properties using these identities.

Example 5.1.

Let *n* and *r* be integers with $0\le r\le n$. Find $\sum_{r=1}^{n} rC_r^n$. Hence find $\sum_{r=1}^{n} r^2 C_r^n$.

Solution.

We have

$$\begin{aligned}\sum_{r=1}^{n} rC_r^n &= \sum_{r=1}^{n} r\left(\frac{n!}{r!(n-r)!}\right)\\ &= \sum_{r=1}^{n} n\left(\frac{(n-1)!}{(r-1)!(n-r)!}\right)\\ &= \sum_{r=1}^{n} n\left(\frac{(n-1)!}{(r-1)!(n-1-(r-1))!}\right)\\ &= \sum_{r=1}^{n} nC_{r-1}^{n-1}\\ &= n2^{n-1}\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{r=1}^{n} r^2 C_r^n &= \sum_{r=1}^{n} r\left(nC_{r-1}^{n-1}\right)\\ &= n\sum_{r=1}^{n} rC_{r-1}^{n-1}\\ &= n\left(\sum_{r=1}^{n}(r-1)C_{r-1}^{n-1} + \sum_{r=1}^{n} C_{r-1}^{n-1}\right)\\ &= n\left(\sum_{r=1}^{n-1} rC_r^{n-1} + \sum_{r=0}^{n-1} C_r^{n-1}\right)\\ &= n\left((n-1)2^{n-2} + 2^{n-1}\right)\\ &= 2^{n-2}\left(n^2 - n + 2\right)\end{aligned}$$

Example 5.2.

Let a, b, r be integers such that $0 \le r \le a \le b$. Using the identity $C_r^{a+b} = C_0^a C_r^b + C_1^a C_{r-1}^b + \cdots + C_r^a C_0^b$, prove that $C_a^{2a} C_b^{2b} \ge \left(C_r^{a+b}\right)^2$.

Solution.

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(C_r^{a+b}\right)^2 &= \left(C_0^a C_r^b + C_1^a C_{r-1}^b + \cdots + C_r^a C_0^b\right)^2\\ &\le \left[\left(C_0^a\right)^2 + \left(C_1^a\right)^2 + \cdots + \left(C_r^a\right)^2\right]\left[\left(C_r^b\right)^2 + \left(C_{r-1}^b\right)^2 + \cdots + \left(C_0^b\right)^2\right]\\ &\le \left[\left(C_0^a\right)^2 + \left(C_1^a\right)^2 + \cdots + \left(C_a^a\right)^2\right]\left[\left(C_0^b\right)^2 + \left(C_1^b\right)^2 + \cdots + \left(C_b^b\right)^2\right]\\ &= C_a^{2a} C_b^{2b}\end{aligned}$$

Example 5.3.

Let *n* and *r* be integers with $0 \le r \le n$. Prove that $C_{r+1}^{n} = C_r^{n-1} + C_r^{n-2} + \cdots + C_r^r$.

Solution.

Using the identity $C_{r+1}^{n} = C_r^{n-1} + C_{r+1}^{n-1}$ repeatedly, we have

$$\begin{aligned}
C_{r+1}^{n} &= C_r^{n-1} + C_{r+1}^{n-1} \\
&= C_r^{n-1} + C_r^{n-2} + C_{r+1}^{n-2} \\
&= C_r^{n-1} + C_r^{n-2} + C_r^{n-3} + C_{r+1}^{n-3} \\
&= \cdots \\
&= C_r^{n-1} + C_r^{n-2} + C_r^{n-3} + \cdots + C_r^{r+1} + C_{r+1}^{r+1} \\
&= C_r^{n-1} + C_r^{n-2} + C_r^{n-3} + \cdots + C_r^{r+1} + C_r^{r}
\end{aligned}$$

6. The Binomial Theorem for Negative and Non-Integral Indices

So far we have been considering the binomial theorem in cases where the power of the expansion is a non-negative integer. In this section we briefly treat the cases where the power is a negative integer or an arbitrary real number.

To begin our discussion we first extend the definition of the binomial coefficients to cases where the upper index is not necessarily a non-negative integer.

Definition 6.1. (Binomial coefficient)

For arbitrary real number *n* and positive integer *r*, we define $C_0^n = 0$ and

$$C_r^n = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!}.$$

Note that this is the same formula as we derived in Section 3. We simply extend it to cases where *n* is not necessarily a non-negative integer.

To extend the binomial theorem to negative and non-integral indices, recall that

$$(x+1)^n = \sum_{r=0}^{n} C_r^n x^r \ .$$

Since $C_r^n = 0$ when $r > n$ (exercise), this can be written as

$$(x+1)^n = \sum_{r=0}^{\infty} C_r^n x^r \ .$$

When n is not a non-negative integer, the above formula gives an infinite series for the expansion of $(x+1)^n$. It can be proved that the series converges for $|x|<1$ and it indeed converges to the expression on the left hand side.

For instance, when $n=-1$, we have

$$(x+1)^{-1} = 1 - x + x^2 - x^3 + \cdots .$$

We know that the (geometric) series on the right converges whenever $|x|<1$, and the formula on the left represents the sum to infinity.

7. Exercises

1. Expand the following using the binomial theorem.

   (a) $(2x-3)^4$ (b) $\left(x-\dfrac{2}{x}\right)^6$

2. Find the coefficient of x^2 in the expansion of

   (a) $(1+x)^4(2-x)^3$ (b) $\left(2x-\dfrac{3}{x^2}\right)^4$

3. Prove that the constant term in the expansion of $\left(x^4-\dfrac{2}{x}\right)^n$ is non-zero if and only if n is a multiple of 5, and the constant term is never negative.

4. Find all positive integers a, b so that in the expression $(1+x)^a+(1+x)^b$, the coefficients of x and x^2 are equal.

5. Let n and r be integers, $0 \le n < r$. Show that $C_r^n = 0$.

6. Prove that for all positive integers n,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = C_3^{2n+1}.$$

7. Explain equation (3) in Section 5 using the combinatorial interpretation of the binomial coefficients.

8. Prove the identity $C_{r-1}^m + C_r^m = C_r^{m+1}$ by using the formula for the binomial coefficients directly.

9. Re-do Example 5.3 by differentiating equation (1) in Section 3.

10. Find the sum

$$C_0^{3n} + C_3^{3n} + C_6^{3n} + \cdots + C_{3n}^{3n}.$$

(Hint: Consider a primitive cube root of unity, i.e. a complex number ω for which $1 + \omega + \omega^2 = 0$ and $\omega \ne 1$.)

11. Let n, r, k be integers such that $0 \le r \le k \le n$.

(a) Prove that $C_k^n C_r^k = C_r^n C_{k-r}^{n-r}$.

(b) Give a combinatorial interpretation of the equality in (a).

12. The sequence of *Catalan numbers* 1, 1, 2, 5, 14, 42, … is given by the formula

$$C_n = \frac{1}{n+1} C_n^{2n} \qquad n = 0, 1, 2, \ldots$$

Prove that $C_n = \dfrac{2(2n-1)}{n+1} C_{n-1}$ for all positive integers n.

13. Let n be a fixed positive integer. Find, in terms of n, the number of n-digit positive integers with the following two properties:

(a) Each digit is either 1, 2 or 3.

(b) The number of 1's is even.